

Knots and Links from Random Projections

By

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Abstract

In this paper we study a model of random knots obtained by fixing a space curve in n -dimensional Euclidean space with $n > 3$, and orthogonally projecting the space curve on to random 3 dimensional subspaces. By varying the space curve we obtain different models of random parametrized knots, and we will study how the expectation value of the curvature changes as a function of the initial parametrized space curve. In the case when the initial data is a pair of space curves, or more generally a pair of manifolds satisfying certain conditions on their dimension, then we obtain models of random links for which we will give methods to compute the second moment of the linking number. As an application of our computations, we will study numerous models of random knots and links, and how to recover these models by appropriately choosing the initial space curves to be projected.

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Chapter 1

Introduction

1.1 Informal Overview and Historical Motivation

Over the past few decades numerous models of random knots and links have been studied and were developed either with a specific application in mind, or with the hope that the model will be sufficiently universal in the sense that it avoids biasing certain knots and links. Models include closed random walks on a lattice, random knots with confinement, random equilateral polygons, random kinematical links, random knots from billiard diagrams, random Fourier knots (for the aforementioned see [2],[4], [7],[10], for example), diagrams sampled from random 4-valent graphs (by Dunfield et. al...see SnapPy documentation: [8]), and the recent Petaluma model studied in [11]. The distributions of the invariants depend on the details of the models, and the difficulty in computing certain statistics resides in the sampling method. Given this vast list, the question of whether or not there is a universal model of random knotting and linking that may contain some of the aforementioned models as special cases remains both ill-posed and unexplored. In this paper we seek to explore this question by introducing a further model of random knots which is sufficiently general that it may be able to encompass some previously studied models of random knots and links.

1.1.1 The Model

In this paper we will consider a model of random knots and links given by first fixing a closed space curve (or pair of closed space curves in the case of links) $\mathbf{r}(t) : \mathbb{S}^1 \rightarrow \mathbb{R}^n$ with $n > 3$, and then choosing a 3 dimensional subspace in \mathbb{R}^n at random to project the curve $\mathbf{r}(t)$ on to. The image under the projection of $\mathbf{r}(t)$ is a parametrized knot in \mathbb{R}^3 for which one may compute a number of quantities associated to the space curve, like the curvature and the writhe. Our hope is to compute the mean and variance of quantities like the curvature, writhe, and linking number with respect to the unique normalized $O(n)$ invariant measure on the Grassmannian of 3-dimensional subspaces in \mathbb{R}^n . The rotational invariance of the measure along with the numerous scaling and multilinearity properties of the aforementioned quantities will allow us to greatly simplify the calculations involved. By choosing different initial curves $\mathbf{r}(t)$ in \mathbb{R}^n , we will obtain different models of knots and links, and moreover we will expect that the distribution of invariants associated with such models to have a crucial dependence on $\mathbf{r}(t)$.

1.1.2 Configuration Space Integrals and Invariants

Since the random knots and links that will be generated with the model described in the previous section will be parametrized, then the sort of quantities that we will consider will be ones that have descriptions as configuration space integrals. In the case of links, one such invariant is the linking number, and when the components of the link are given by non-intersecting differentiable curves, γ_1 and γ_2 , given by the parametrizations $\gamma_1 = \mathbf{r}_1(t)$ and $\gamma_2 = \mathbf{r}_2(s)$, then the linking number may be computed using the Gauss linking integral:

$$\text{Lk}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{(s,t) \in \mathbb{T}^2} \frac{(\dot{\mathbf{r}}_1(t) \times \dot{\mathbf{r}}_2(s)) \cdot (\mathbf{r}_2(s) - \mathbf{r}_1(t))}{\|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^3} dA \quad (1.1.1)$$

where $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is the torus where each factor is the interval $[0, 2\pi]$ with its endpoints identified, and $dA = dsdt$. As motivation, we will use the model described in the previous section to find the expected value of a closely related quantity, the average inter-crossing number, which we will denote ICN (see [9] for further details):

$$\text{ICN}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{(s,t) \in \mathbb{T}^2} \frac{|(\dot{\mathbf{r}}_1(t) \times \dot{\mathbf{r}}_2(s)) \cdot (\mathbf{r}_2(s) - \mathbf{r}_1(t))|}{\|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^3} dA$$

To do so, let $\gamma_1 = \mathbf{r}_1(t)$ and $\gamma_2 = \mathbf{r}_2(s)$ be differentiable closed space curves in \mathbb{R}^4 and define $\mathbf{v} = (v_1, v_2, v_3, v_4)$ and $d\mathbf{v} = \exp(-\frac{1}{2}(v_1^2 + v_2^2 + v_3^2 + v_4^2)) dv_1 dv_2 dv_3 dv_4$. This way, we have that:

$$\langle \text{ICN} \rangle = \frac{1}{4\pi(2\pi)^2} \int_{\mathbf{v} \in \mathbb{R}^4} \int_{\mathbb{T}^2} \frac{|\text{Det}[\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t), \frac{\mathbf{v}}{\|\mathbf{v}\|}]|}{\|\text{proj}_{\mathbf{v}^\perp}(\mathbf{r}_2(s) - \mathbf{r}_1(t))\|^3} dsdt d\mathbf{v}$$

where $\text{proj}_{\mathbf{v}^\perp}(\mathbf{r}_2(s) - \mathbf{r}_1(t))$ denotes the orthogonal projection on to the orthogonal complement of $\frac{\mathbf{v}}{\|\mathbf{v}\|}$. In order to emphasize the dependence on the initial data $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$, we will reverse the order of integration:

$$\langle \text{ICN} \rangle = \frac{1}{4\pi(2\pi)^2} \int_{\mathbb{T}^2} \int_{\mathbf{v} \in \mathbb{R}^4} \frac{|\text{Det}[\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t), \frac{\mathbf{v}}{\|\mathbf{v}\|}]|}{\|\text{proj}_{\mathbf{v}^\perp}(\mathbf{r}_2(s) - \mathbf{r}_1(t))\|^3} d\mathbf{v} dA$$

Next, given vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^n$ define a matrix A by making the vectors \mathbf{a}_i the columns, which from now on we will denote by $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$, and then define the function:

$$I_{\langle \text{ICN} \rangle}(A) = \frac{1}{(2\pi)^2} \int_{\mathbf{v} \in \mathbb{R}^4} \frac{|\text{Det}[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \frac{\mathbf{v}}{\|\mathbf{v}\|}]|}{\|\text{proj}_{\mathbf{v}^\perp}(\mathbf{a}_3)\|^3} d\mathbf{v} \quad (1.1.2)$$

1.1. Informal Overview and Historical Motivation

where $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \frac{\mathbf{v}}{\|\mathbf{v}\|}] \in \text{Mat}_{4,4}$. Given the definition in (1.1.2), we may simplify the formula for $\langle \text{ICN} \rangle$ as:

$$\langle \text{ICN} \rangle = \frac{1}{4\pi} \int_{\mathbb{T}^2} I_{\langle \text{ICN} \rangle}([\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t)]) ds dt$$

It will be shown in the following sections that $I_{\langle \text{ICN} \rangle}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3])$ has an especially nice form, and is given by:

$$I_{\langle \text{ICN} \rangle}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]) = \frac{\sqrt{\text{Det}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]^T [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3])}}{\|\mathbf{a}_3\|^3}, \quad (1.1.3)$$

so that:

$$\begin{aligned} \langle \text{ICN} \rangle &= \frac{1}{4\pi} \int_{\mathbb{T}^2} I_{\langle \text{ICN} \rangle}([\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t)]) \\ &= \frac{1}{4\pi} \int_{\mathbb{T}^2} \frac{\sqrt{\text{Det}([\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t)]^T [\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t)])}}{\|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^3} ds dt \end{aligned} \quad (1.1.4)$$

For the most part, the goal of this work will be to both find closed forms and understand the analytic properties of functions like $I_{\langle \text{ICN} \rangle}([\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t)])$. By performing this procedure of reversing the order of integration, and then integrating over all projections, we may then find bounds on $\langle \text{ICN} \rangle$ in terms of the parametrization of the initial space curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$. The computation leading to (1.1.4) will be completed in the second chapter, along with a similar analysis for $\langle \kappa(C) \rangle$, where $\kappa(C)$ is the total curvature of a knot (the total curvature case is not a new result, and was already discovered in a classic paper by [12], [16], and can be found in recent papers such as [19]).

Since $\kappa(C)$ and $\langle \text{ICN} \rangle$ are not invariants, but are only bounds on invariants of knots and links, then it will be a bit more interesting to consider instead $\langle \text{Lk}^2 \rangle$ and the corresponding function $I_{\langle \text{Lk}^2 \rangle}(A, A')$ obtained from reversing the order of integration as was demonstrated above. Given two space curves γ_1 and γ_2 with parametrizations $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$, then we may compute the value of $\text{Lk}^2(\gamma_1, \gamma_2)$ from the Gauss linking integral by computing:

$$\begin{aligned} \text{Lk}^2(\gamma_1, \gamma_2) &= \\ \frac{1}{16\pi^2} \int_{(s,t) \in \mathbb{T}^2} \int_{(s',t') \in \mathbb{T}^2} &\frac{\text{Det}[\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t)] \text{Det}[\dot{\mathbf{r}}_1(t'), \dot{\mathbf{r}}_2(s'), \mathbf{r}_2(s') - \mathbf{r}_1(t')]}{\|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^3 \|\mathbf{r}_2(s') - \mathbf{r}_1(t')\|^3} ds dt ds' dt' \end{aligned}$$

Figure 1.1.2 depicts the configurations of pairs of points that we will typically want to integrate over in order to find the value of $\text{Lk}^2(\gamma_1, \gamma_2)$.

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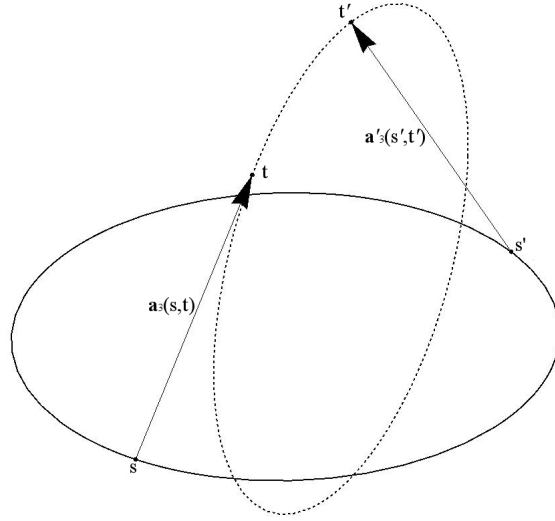


Figure 1.1.1: The geometric configuration for computing Lk^2 for the closed space curves $\mathbf{r}_1(s)$ and $\mathbf{r}_2(t)$ where $\mathbf{a}_3(s, t) = \mathbf{r}_2(s) - \mathbf{r}_1(t)$ and $\mathbf{a}_3(s', t') = \mathbf{r}_2(s') - \mathbf{r}_1(t')$

Obtaining $I_{\langle \text{Lk}^2 \rangle}(A, A')$ in this case is considerably more difficult, and will occupy the third chapter, which is the main part of this work. The computation involved after switching the order of integration is summarized with the following lemma:

Lemma 1.1.1. *Given matrices $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$, $A' = [\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3] \in \text{Mat}_{4,3}$ and the function*

$$I_{\langle \text{Lk}^2 \rangle}(A, A') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \frac{\text{Det}([A \frac{\mathbf{v}}{\|\mathbf{v}\|}]) \text{Det}([A' \frac{\mathbf{v}}{\|\mathbf{v}\|}])}{\|\text{proj}_{\mathbf{v}^\perp}(\mathbf{a}_3)\|^3 \|\text{proj}_{\mathbf{v}^\perp}(\mathbf{a}'_3)\|^3} e^{-\|\mathbf{v}\|^2/2} d\mathbf{v}$$

where $[A \frac{\mathbf{v}}{\|\mathbf{v}\|}]$ denotes a new matrix in $\text{Mat}_{4,4}$ with the column $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ appended to the matrix A , then we have

$$\begin{aligned} \frac{\pi}{2} I_{\langle \text{Lk}^2 \rangle}(A, A') = & \frac{(\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2) \text{Det}(A^T A') + (\mathbf{a}_3 \cdot \mathbf{a}'_3) \text{Det}([\mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3]) \text{Det}([\mathbf{a}'_3, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3])}{\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 (\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2)^{3/2}} \end{aligned}$$

It will be seen throughout the course of the proof of the above lemma that the requirement that the initial space curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ be embedded in \mathbb{R}^4 serves to simplify the integration over $Gr(4, 3)$ since the subspaces we are integrating over are codimension 1 so that $Gr(4, 3) \cong \mathbb{S}^3$. This restrictive assumption will be removed in the second half of chapter 3 by introducing the Stiefel manifold of orthonormal 3-frames in \mathbb{R}^n , denoted $V_{n,3}$, and using a decomposition of its unique normalized $O(n)$ measure coupled with the previous result in order to find $\langle \text{Lk}^2 \rangle$ for any pair of curves in \mathbb{R}^n , for n sufficiently large.

1.2 Summary of the Main Results

Theorem 1.2.1. *Given two closed, differentiable, non-intersecting space curves $\gamma_1, \gamma_2 : \mathbb{S}^1 \rightarrow \mathbb{R}^n$ with parametrizations $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$, and let $\text{proj}_U(\gamma_1), \text{proj}_U(\gamma_2)$ denote the orthogonal projections of γ_1 and γ_2 on to the 3 dimensional subspace spanned by the columns of U , then the expected value of $\text{ICN}(\text{proj}_U(\gamma_1), \text{proj}_U(\gamma_2))$, denoted $\langle \text{ICN} \rangle$, averaged over all orthogonal projections to 3 dimensional subspaces with respect to the unique normalized $O(n)$ -invariant measure on $\text{Gr}(n, 3)$, is given by the following integral:*

$$\langle \text{ICN} \rangle = \frac{1}{4\pi} \int_{(s,t) \in \mathbb{T}^2} I_{\langle \text{ICN} \rangle}(s, t) ds dt$$

where

$$I_{\langle \text{ICN} \rangle}(s, t) = C_{\langle \text{ICN} \rangle} \frac{\sqrt{\text{Det}\left(\left[\frac{d\mathbf{r}_1(t)}{dt}, \frac{d\mathbf{r}_2(s)}{ds}, \mathbf{r}_2(s) - \mathbf{r}_1(t)\right]^T \left[\frac{d\mathbf{r}_1(t)}{dt}, \frac{d\mathbf{r}_2(s)}{ds}, \mathbf{r}_2(s) - \mathbf{r}_1(t)\right]\right)}}{\|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^3},$$

where $C_{\langle \text{ICN} \rangle}$ is a constant.

After proving the above theorem we then focus on the computation of $\langle \text{ICN} \rangle$:

Theorem 1.2.2. *Let $\mathbf{r}_1(t) : \mathbb{S}^1 \rightarrow \mathbb{R}^{2n+1}$ and $\mathbf{r}_2(s) : \mathbb{S}^1 \rightarrow \mathbb{R}^{2n+1}$ be given by:*

$$\begin{aligned} \mathbf{r}_1(t) &= c_0 \mathbf{e}_0 + \sum_{k=1}^n (c_k \cos(kt) \mathbf{e}_{2k-1} + c_k \sin(kt) \mathbf{e}_{2k}) \\ &= (c_0, c_1 \cos(t), c_1 \sin(t), c_2 \cos(2t), c_2 \sin(2t), \dots, c_n \cos(nt), c_n \sin(nt)), \text{ and} \\ \mathbf{r}_2(s) &= d_0 \mathbf{e}_0 + \sum_{k=1}^n (d_k \cos(ks) \mathbf{e}_{2k-1} + d_k \sin(ks) \mathbf{e}_{2k}) \\ &= (d_0, d_1 \cos(s), d_1 \sin(s), d_2 \cos(2s), d_2 \sin(2s), \dots, d_n \cos(ns), d_n \sin(ns)), \end{aligned}$$

where $\{\mathbf{e}_i\}_{i=1}^n$ are the standard basis vectors, then $\langle \text{ICN} \rangle$ is finite and satisfies the following bound:

$$\langle \text{ICN} \rangle \leq C_{\langle \text{ICN} \rangle} \frac{\sqrt{(\sum_{j=0}^n j^2 c_j^2)(\sum_{j=0}^n j^2 d_j^2)}}{\min_{(s,t) \in \mathbb{T}^2} \|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^2}$$

In chapter 3 we move on to proving the main theorem of the paper concerning the second moment of the linking number.

Theorem 1.2.3. *Given two closed, differentiable, non-intersecting space curves $\gamma_1, \gamma_2 : \mathbb{S}^1 \rightarrow \mathbb{R}^n$ with parametrizations $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$, and let $\text{proj}_U(\gamma_1), \text{proj}_U(\gamma_2)$ denote the orthogonal projections of γ_1 and γ_2 on to the 3 dimensional subspace spanned by the columns of U , then the expected value of $\text{Lk}^2(\text{proj}_U(\gamma_1), \text{proj}_U(\gamma_2))$, denoted $\langle \text{Lk}^2 \rangle$, averaged over all orthogonal projections to 3 dimensional subspaces with respect to the unique normalized $O(n)$ -invariant*

1.2. Summary of the Main Results

measure on $Gr(n, 3)$, is given by the following integral:

$$\langle \text{Lk}^2 \rangle = \frac{1}{16\pi^2} \int_{(s,t) \in \mathbb{T}^2} \int_{(s',t') \in \mathbb{T}^2} I_{\langle \text{Lk}^2 \rangle}(A(s, t), A'(s', t')) ds dt ds' dt'$$

where

$$\begin{aligned} \frac{\pi}{2} I_{\langle \text{Lk}^2 \rangle}(A(s, t), A'(s', t')) = \\ \frac{(\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2) \text{Det}(A^T A') + (\mathbf{a}_3 \cdot \mathbf{a}'_3) \text{Det}([\mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3]) \text{Det}([\mathbf{a}'_3, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3])}{\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 (\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2)^{3/2}} \end{aligned}$$

and

$$\begin{aligned} \mathbf{a}_1 &= \dot{\mathbf{r}}_1(t), \mathbf{a}_2 = \dot{\mathbf{r}}_2(s), \mathbf{a}_3 = \mathbf{r}_2(s) - \mathbf{r}_1(t) \\ \mathbf{a}'_1 &= \dot{\mathbf{r}}_1(t'), \mathbf{a}'_2 = \dot{\mathbf{r}}_2(s'), \mathbf{a}'_3 = \mathbf{r}_2(s') - \mathbf{r}_1(t') \\ A &= [\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t)] \\ A' &= [\dot{\mathbf{r}}_1(t'), \dot{\mathbf{r}}_2(s'), \mathbf{r}_2(s') - \mathbf{r}_1(t')] \end{aligned}$$

After integrating over all the projections, we then turn our attention to integrating over the configuration space.

Theorem 1.2.4. *With the definitions above, let $v_1 = \max_{t \in \mathbb{S}^1} \|\dot{\mathbf{r}}_1(t)\|$, $v_2 = \max_{s \in \mathbb{S}^1} \|\dot{\mathbf{r}}_2(s)\|$, $k = \min_{(s,t) \in \mathbb{T}^2} \|\mathbf{r}_2(t) - \mathbf{r}_1(s)\|$ and*

$$C = \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{\|\mathbf{a}_3(s, t)\|^2 \|\mathbf{a}'_3(s', t')\|^2 - (\mathbf{a}_3(s, t) \cdot \mathbf{a}'_3(s', t'))^2}} ds dt ds' dt',$$

then

$$\langle \text{Lk}^2 \rangle \leq \frac{1}{(4\pi)^2} \frac{4C v_1^2 v_2^2}{\pi k^2}$$

That is, we obtain a bound on the second moment of the linking number provided the initial space curves are chosen so that v_1, v_2 and C are finite. In the fourth chapter we will prove theorems very analogous to theorems in chapter 3 concerning the second moment of linking numbers associated to manifolds as defined in [18]. Lastly, in order to show the applicability of some of our results, in the penultimate chapter we consider a model that is very similar to the model in [11].

Chapter 2

Expectation Values

2.1 Background

In the first part of this chapter we will find the expectation value of $\langle \text{ICN} \rangle$ when we fix a pair of space curves in \mathbb{R}^4 and project the curves randomly on to 3 dimensional subspaces. As discussed in the introduction, this will require two steps, the first will be integrating over all 3 dimensional subspaces, and then integrating that result over the configuration space of two points on distinct components of a link. The first step can be done exactly, whereas the second will be approached by making specific assumptions about the initial data. In the second part of this chapter we will find the expected curvature when we fix one space curve in \mathbb{R}^n , and project the curve randomly on to 3 dimensional subspaces, where we pick random subspaces by sampling the spans of the columns of Gaussian random matrices in $\text{Mat}_{n,3}$.

2.2 Integration Over 3-Dimensional Subspaces

2.2.1 Average Intercrossing Number: $\langle \text{ICN} \rangle$

We start with the following lemma:

Lemma 2.2.1. *Given a matrix $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \in \text{Mat}_{4,3}$ and the function*

$$I_{\langle \text{ICN} \rangle}(A) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \frac{|\text{Det}([A \frac{\mathbf{v}}{\|\mathbf{v}\|}])|}{\|\text{proj}_{\mathbf{v}^\perp}(\mathbf{a}_3)\|^3} e^{-\|\mathbf{v}\|^2/2} d\mathbf{v}$$

where $[A \frac{\mathbf{v}}{\|\mathbf{v}\|}]$ denotes a new matrix in $\text{Mat}_{4,4}$ with the column $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ appended to the matrix A , then we have

$$I_{\langle \text{ICN} \rangle}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]) = \frac{\sqrt{\text{Det}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]^T [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3])}}{\|\mathbf{a}_3\|^3},$$

2.2. Integration Over 3-Dimensional Subspaces

PROOF We first observe the following properties of $I_{\langle \text{ICN} \rangle}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3])$:

$$I_{\langle \text{ICN} \rangle} \left(A \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \right) = \frac{|k_1| |k_2|}{|k_3|^2} I_{\langle \text{ICN} \rangle}(A) \quad (2.2.1)$$

$$I_{\langle \text{ICN} \rangle} \left(A \begin{pmatrix} 1 & 0 & 0 \\ \frac{-\mathbf{a}_1 \cdot (\mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{a}_3}{\|\mathbf{a}_3\|^2} \mathbf{a}_3)}{\|(\mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{a}_3}{\|\mathbf{a}_3\|^2} \mathbf{a}_3)\|} & 1 & 0 \\ \frac{-\mathbf{a}_1 \cdot \mathbf{a}_3}{\|\mathbf{a}_3\|^2} & \frac{-\mathbf{a}_2 \cdot \mathbf{a}_3}{\|\mathbf{a}_3\|^2} & 1 \end{pmatrix} \right) = I_{\langle \text{ICN} \rangle}(A) \quad (2.2.2)$$

Using (2.2.2) we may make the columns orthogonal, and then using (2.2.1) we may further make them orthonormal. Next, using the fact that the measure $e^{-\|\mathbf{v}\|^2/2} d\mathbf{v}$ is $O(4)$ invariant, we may take the orthonormal set of vectors to coincide with the standard basis vectors, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ so that $I_{\langle \text{ICN} \rangle}(A)$ may be rewritten as:

$$\begin{aligned} I_{\langle \text{ICN} \rangle}(A) &= \frac{f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)}{(2\pi)^2} \int_{\mathbf{v} \in \mathbb{R}^4} \frac{|\text{Det}([\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3 \ \frac{\mathbf{v}}{\|\mathbf{v}\|}])|}{\|\text{proj}_{\mathbf{v}^\perp}(\mathbf{a}_3)\|^3} \exp(-\frac{\|\mathbf{v}\|^2}{2}) d\mathbf{v} \\ &= f(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) I_{|\text{Lk}|}([\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]), \end{aligned}$$

where

$$\begin{aligned} f([\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]) &= \frac{\|\mathbf{a}_1 - \frac{\mathbf{a}_1 \cdot (\mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{a}_3}{\|\mathbf{a}_3\|^2} \mathbf{a}_3)}{\|(\mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{a}_3}{\|\mathbf{a}_3\|^2} \mathbf{a}_3)\|} (\mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{a}_3}{\|\mathbf{a}_3\|^2} \mathbf{a}_3) - \frac{\mathbf{a}_1 \cdot \mathbf{a}_3}{\|\mathbf{a}_3\|^2} \mathbf{a}_3\| \|\mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{a}_3}{\|\mathbf{a}_3\|^2} \mathbf{a}_3\| \|\mathbf{a}_3\|}{\|\mathbf{a}_3\|^3} \\ &= \frac{\sqrt{\text{Det}(A^T A)}}{\|\mathbf{a}_3\|^3}, \text{ and} \end{aligned}$$

Now we will compute the integral:

$$\begin{aligned} I_{\langle \text{ICN} \rangle}(A) &= \frac{1}{(2\pi)^2} f([\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]) \int_{\mathbb{R}^4} \frac{|\text{Det}([\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3 \ \frac{\mathbf{v}}{\|\mathbf{v}\|}])|}{\|\text{proj}_{\mathbf{v}^\perp}(\mathbf{e}_3)\|^3} \exp(-\frac{\|\mathbf{v}\|^2}{2}) d\mathbf{v} \\ &= \frac{1}{(2\pi)^2} f([\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]) \int_{\mathbb{R}^4} \frac{|v_4/\|\mathbf{v}\||}{\|\text{proj}_{\mathbf{v}^\perp}(\mathbf{e}_3)\|^3} \exp(-\frac{\|\mathbf{v}\|^2}{2}) d\mathbf{v} \\ &= \frac{1}{(2\pi)^2} f([\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]) \int_{\mathbb{R}^4} \frac{|v_4/\|\mathbf{v}\||}{(\frac{\|\mathbf{v}\|^2 - v_3^2}{\|\mathbf{v}\|^2})^{3/2}} \exp(-\frac{\|\mathbf{v}\|^2}{2}) d\mathbf{v} = \\ &= \frac{1}{(2\pi)^2} f([\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]) \int_{\mathbb{R}^4} \frac{\|\mathbf{v}\|^2 |v_4|}{(\|\mathbf{v}\|^2 - v_3^2)^{3/2}} \exp(-\frac{\|\mathbf{v}\|^2}{2}) d\mathbf{v} = f([\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]) \end{aligned}$$

□

2.2. Integration Over 3-Dimensional Subspaces

Given two space curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ in \mathbb{R}^4 , if we set

$$\begin{aligned}\mathbf{a}_1(t) &= \frac{d\mathbf{r}_1(t)}{dt} \\ \mathbf{a}_2(s) &= \frac{d\mathbf{r}_2(s)}{ds} \\ \mathbf{a}_3(t, s) &= \mathbf{r}_2(s) - \mathbf{r}_1(t) \text{ and} \\ A(s, t) &= [\mathbf{a}_1(t), \mathbf{a}_2(s), \mathbf{a}_3(s, t)]\end{aligned}$$

in the previous lemma, then we will have the following lemma:

Lemma 2.2.2. *Given two differentiable, non-intersecting space curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ in \mathbb{R}^4 , then the value of ICN averaged over all orthogonal projections to 3 dimensional subspaces with respect to the unique normalized $O(4)$ -invariant measure on $Gr(4, 3)$, is given by the following integral:*

$$\langle \text{ICN} \rangle = \frac{1}{4\pi} \int_{(s,t) \in \mathbb{T}^2} I_{\langle \text{ICN} \rangle}(A(s, t)) ds dt$$

PROOF Consider the integral, $I_{\langle \text{ICN} \rangle}(A)$, from the previous lemma. We may integrate over $\mathbf{v} = \{v_1, v_2, v_3, v_4\}$ by changing to 4-dimensional spherical coordinates where

$$\begin{aligned}v_1 &= r \cos(\phi_1) \\ v_2 &= r \sin(\phi_1) \cos(\phi_2) \\ v_3 &= r \sin(\phi_1) \sin(\phi_2) \cos(\phi_3) \\ v_4 &= r \sin(\phi_1) \sin(\phi_2) \sin(\phi_3)\end{aligned}$$

so that $e^{-\|\mathbf{v}\|^2/2} d\mathbf{v} = e^{-r^2/2} r^3 \sin^2(\phi_1) \sin(\phi_2) dr d\phi_1 d\phi_2 d\phi_3$ and $r \in [0, \infty)$, $\phi_1 \in [0, \pi]$, $\phi_2 \in [0, \pi]$, $\phi_3 \in [0, 2\pi)$. Since the function we are integrating against $e^{-r^2/2} r^3 \sin^2(\phi_1) \sin(\phi_2) dr d\phi_1 d\phi_2 d\phi_3$ is invariant under scaling \mathbf{v} , then we may integrate out the radial coordinate r in order to obtain an integral over a 3-sphere, and the result follows from the previous lemma since $\mathbb{S}^3 \cong Gr(4, 3)$ and since $Gr(4, 3)$ has a unique normalized $O(4)$ invariant measure on it. \square

Using the techniques in [22] and that we will develop in the proof of 1.2.3, it is straightforward to show that:

Theorem 1.2.1. *Given two closed, differentiable, non-intersecting space curves $\gamma_1, \gamma_2 : \mathbb{S}^1 \rightarrow \mathbb{R}^n$ with parametrizations $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$, and let $\text{proj}_U(\gamma_1), \text{proj}_U(\gamma_2)$ denote the orthogonal projections of γ_1 and γ_2 on to the 3 dimensional subspace spanned by the columns of U , then the expected value of $\text{ICN}(\text{proj}_U(\gamma_1), \text{proj}_U(\gamma_2))$, denoted $\langle \text{ICN} \rangle$, averaged over all orthogonal projections to 3 dimensional subspaces with respect to the unique normalized $O(n)$ -invariant measure on $Gr(n, 3)$, is given by the following integral:*

$$\langle \text{ICN} \rangle = \frac{1}{4\pi} \int_{(s,t) \in \mathbb{T}^2} I_{\langle \text{ICN} \rangle}(s, t) ds dt$$

2.2. Integration Over 3-Dimensional Subspaces

where

$$I_{\langle \text{ICN} \rangle}(s, t) = C_{\langle \text{ICN} \rangle} \frac{\sqrt{\text{Det}\left(\left[\frac{d\mathbf{r}_1(t)}{dt}, \frac{d\mathbf{r}_2(s)}{ds}, \mathbf{r}_2(s) - \mathbf{r}_1(t)\right]^T \left[\frac{d\mathbf{r}_1(t)}{dt}, \frac{d\mathbf{r}_2(s)}{ds}, \mathbf{r}_2(s) - \mathbf{r}_1(t)\right]\right)}}{\|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^3},$$

where $C_{\langle \text{ICN} \rangle}$ is a constant.

PROOF We will focus only on integrating over $\text{Mat}_{n,3}$, since the integration over $\text{Gr}(n, 3)$ follows easily from the following computation and the discussion in the next chapter. Define that:

$$I_{\langle \text{ICN} \rangle}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]) = \frac{1}{(2\pi)^{3n/2}} \int_{U \in \text{Mat}_{n,3}} \frac{\sqrt{\text{Det}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]^T U (U^T U)^{-1} U^T [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3])}}{(\mathbf{a}_3^T U (U^T U)^{-1} U^T \mathbf{a}_3)^{3/2}} dU,$$

where we think of $U \in \text{Mat}_{n,3}$ as a matrix with columns $\mathbf{u}_1, \mathbf{u}_2$ and \mathbf{u}_3 , and

$$dU = \exp\left(-\frac{1}{2}\text{Tr}(U^T U)\right) d\mathbf{u}_1 d\mathbf{u}_2 d\mathbf{u}_3$$

Next, apply the Gram-Schmidt process to the ordered set of vectors $\{\mathbf{a}_3, \mathbf{a}_2, \mathbf{a}_1\}$ to obtain the orthonormal set $\{\mathbf{q}_3, \mathbf{q}_2, \mathbf{q}_1\}$, so that we have:

$$\begin{aligned} I_{\langle \text{ICN} \rangle}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]) &= \frac{1}{(2\pi)^{3n/2}} \frac{\text{Det}(A^T A)^{1/2}}{\|\mathbf{a}_3\|^3} \int_{U \in \text{Mat}_{n,3}} \frac{\sqrt{\text{Det}([\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]^T U (U^T U)^{-1} U^T [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3])}}{(\mathbf{q}_3^T U (U^T U)^{-1} U^T \mathbf{q}_3)^{3/2}} dU \\ &= \frac{\text{Det}(A^T A)^{1/2}}{\|\mathbf{a}_3\|^3} I_{\langle \text{ICN} \rangle}([\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]), \end{aligned}$$

where in the above $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$. Since the measure dU is $O(n)$ invariant, then we may take the set of vectors $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ to coincide with the first 3 standard basis vectors: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The result follows by setting $C_{\langle \text{ICN} \rangle} = I_{\langle \text{ICN} \rangle}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ \square

Remark 2.2.3. Notice that if we considered $\langle \text{Lk} \rangle$ instead, the integral would have vanished.

2.2.2 Total Curvature

For pedagogical purposes, we will apply a similar analysis to find the expected total curvature of a knot. For the most part, the following is due to Fary [12], Milnor [16], and appears in a more generalized context by Sullivan [19]. In this section, we will simply show that the aforementioned arguments easily fit in to our framework discussed previously. To do so, recall that if we are given a twice continuously differentiable space curve, C , with parametrization $\mathbf{r}(t) : \mathbb{S}^1 \rightarrow \mathbb{R}^3$, we may compute the total curvature, $\kappa(C)$, using a Gauss-like integral:

$$\kappa(C) = \int_0^{2\pi} \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{\|\mathbf{r}'(t)\|^2} dt = \int_0^{2\pi} \frac{\sqrt{\|\mathbf{r}'(t)\|^2 \|\mathbf{r}''(t)\|^2 - (\mathbf{r}'(t) \cdot \mathbf{r}''(t))^2}}{\|\mathbf{r}'(t)\|^2} dt,$$

2.2. Integration Over 3-Dimensional Subspaces

First we will prove a lemma analogous to 2.2.1:

Lemma 2.2.4. *Given a matrix $A = [\mathbf{a}_1, \mathbf{a}_2] \in \text{Mat}_{n,2}$ and the function*

$$I_{\kappa(\mathbf{C})}(A) = \frac{1}{(2\pi)^{3n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\text{Det}(A^T U (U^T U)^{-1} U^T A)|^{1/2}}{(\mathbf{a}_1^T U (U^T U)^{-1} U^T \mathbf{a}_1)^2} dU$$

where U denotes the matrix $U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ and $dU = \exp(-\frac{1}{2}(\|\mathbf{u}_1\|^2 + \|\mathbf{u}_2\|^2 + \|\mathbf{u}_3\|^2)) d\mathbf{u}_1 d\mathbf{u}_2 d\mathbf{u}_3$, then

$$I_{\kappa(\mathbf{C})}(A) = k \frac{\sqrt{\|\mathbf{a}_1\|^2 \|\mathbf{a}_2\|^2 - (\mathbf{a}_1 \cdot \mathbf{a}_2)^2}}{\|\mathbf{a}_1\|^2}$$

where k is a constant.

PROOF As in the proof of lemma 2.2.1, we may use the numerous multilinearity and scaling properties to write $I_{\kappa(\mathbf{C})}(A)$ as:

$$I_{\kappa(\mathbf{C})}(A) = \frac{1}{(2\pi)^{3n/2}} \frac{\sqrt{|\text{Det}(A^T A)|}}{\|\mathbf{a}_1\|^2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\text{Det}(\tilde{U}(U^T U)^{-1} \tilde{U}^T)|^{1/2}}{(\widetilde{U_{(1)}}(U^T U)^{-1} \widetilde{U_{(1)}}^T)^2} dU$$

where U is an n by 3 matrix with columns $\mathbf{u}_1, \mathbf{u}_2$ and \mathbf{u}_3 , and

$$\tilde{U} = \begin{pmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \end{pmatrix}$$

and

$$\widetilde{U_{(1)}} = \begin{pmatrix} u_{11} & u_{21} & u_{31} \end{pmatrix},$$

where u_{ij} is the j^{th} entry of the vector \mathbf{u}_i .

The result follows by setting $k = \frac{1}{(2\pi)^{3n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\text{Det}(\tilde{U}(U^T U)^{-1} \tilde{U}^T)|^{1/2}}{(\widetilde{U_{(1)}}(U^T U)^{-1} \widetilde{U_{(1)}}^T)^2} dU$. \square

The following is essentially due to Fary and Milnor:

Theorem 2.2.5. *Let C be a twice-differentiable space curve with parametrization $\mathbf{r}(t) : \mathbb{S}^1 \rightarrow \mathbb{R}^n$ and $\text{proj}_U(\mathbf{r}(t))$ be the orthogonal projection of $\mathbf{r}(t)$ to the 3-dimensional subspace spanned by the columns of U , then the value of the total curvature, $\kappa(\text{proj}_U(\mathbf{r}(t)))$, averaged over all projections to 3 dimensional subspaces with respect to the Gaussian measure on $\text{Mat}_{n,3}$ is given by the integral:*

$$\langle \kappa(\mathbf{C}) \rangle = \int_{t \in \mathbb{T}^1} I_{\kappa(\mathbf{C})}(t) dt$$

where

$$I_{\kappa(\mathbf{C})}(t) = \frac{\sqrt{\|\mathbf{r}'(t)\|^2 \|\mathbf{r}''(t)\|^2 - (\mathbf{r}'(t) \cdot \mathbf{r}''(t))^2}}{\|\mathbf{r}'(t)\|^2}$$

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PROOF Given the twice differentiable curve $\mathbf{r}(t) : \mathbb{S}^1 \rightarrow \mathbb{R}^n$, we set

$$\begin{aligned}\mathbf{a}_1(t) &= \frac{d\mathbf{r}_1(t)}{dt} \\ \mathbf{a}_2(t) &= \frac{d^2\mathbf{r}_1(t)}{dt^2} \\ A(t) &= [\mathbf{a}_1(t), \mathbf{a}_2(t)]\end{aligned}$$

This way we will have that:

$$\langle \kappa(C) \rangle = \int_0^{2\pi} I_{\kappa(C)}(A(t)) dt$$

Using the previous lemma we have that:

$$\langle \kappa(C) \rangle = \int_0^{2\pi} k \frac{\sqrt{\|\mathbf{a}_1(t)\|^2 \|\mathbf{a}_2(t)\|^2 - (\mathbf{a}_1(t) \cdot \mathbf{a}_2(t))^2}}{\|\mathbf{a}_1(t)\|^2} dt$$

Since the above holds for any $\mathbf{r}(t)$, then in particular we may take $\mathbf{r}(t) = \cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_2$, so that $\langle \kappa(C) \rangle = 2\pi$ since the image of almost all projections will be a planar ellipse. It is also easy to compute that $\int_0^{2\pi} k \frac{\sqrt{\|\mathbf{a}_1(t)\|^2 \|\mathbf{a}_2(t)\|^2 - (\mathbf{a}_1(t) \cdot \mathbf{a}_2(t))^2}}{\|\mathbf{a}_1(t)\|^2} dt = 2\pi$, so that we see:

$$\frac{1}{(2\pi)^{3n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\text{Det}(\tilde{U}(U^T U)^{-1} \tilde{U}^T)|^{1/2}}{(U_{(1)}^T (U^T U)^{-1} U^T U_{(1)}^T)^2} \exp^{-\frac{1}{2}(\|\mathbf{u}_1\|^2 + \|\mathbf{u}_2\|^2 + \|\mathbf{u}_3\|^2)} d\mathbf{u}_1 d\mathbf{u}_2 d\mathbf{u}_3 = 1$$

and therefore

$$\langle \kappa(C) \rangle = \int_0^{2\pi} \frac{\sqrt{\|\mathbf{r}'(t)\|^2 \|\mathbf{r}''(t)\|^2 - (\mathbf{r}'(t) \cdot \mathbf{r}''(t))^2}}{\|\mathbf{r}'(t)\|^2} dt \quad (2.2.3)$$

□

That is, the expected total curvature of one of the knot projections is simply the total curvature of the space curve C that we are randomly projecting to lower dimensions.

2.3 Integration Over the Configuration Space

Up to this point we have only considered the integration over all the 3 dimensional subspaces, and now we will consider how to specify the initial data, so as to bound $\langle \kappa(C) \rangle$ and $\langle \text{ICN} \rangle$ and therefore bound the complexity of the curvature and the average inter-crossing number. We will now prove the following theorem:

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Theorem 1.2.2. *Let $\mathbf{r}_1(t) : \mathbb{S}^1 \rightarrow \mathbb{R}^{2n+1}$ and $\mathbf{r}_2(s) : \mathbb{S}^1 \rightarrow \mathbb{R}^{2n+1}$ be given by:*

$$\begin{aligned}\mathbf{r}_1(t) &= c_0 \mathbf{e}_0 + \sum_{k=1}^n (c_k \cos(kt) \mathbf{e}_{2k-1} + c_k \sin(kt) \mathbf{e}_{2k}) \\ &= (c_0, c_1 \cos(t), c_1 \sin(t), c_2 \cos(2t), c_2 \sin(2t), \dots, c_n \cos(nt), c_n \sin(nt)), \text{ and} \\ \mathbf{r}_2(s) &= d_0 \mathbf{e}_0 + \sum_{k=1}^n (d_k \cos(ks) \mathbf{e}_{2k-1} + d_k \sin(ks) \mathbf{e}_{2k}) \\ &= (d_0, d_1 \cos(s), d_1 \sin(s), d_2 \cos(2s), d_2 \sin(2s), \dots, d_n \cos(ns), d_n \sin(ns))\end{aligned}$$

then $\langle \text{ICN} \rangle$ is finite and satisfies the following bound:

$$\langle \text{ICN} \rangle \leq C_{\langle \text{ICN} \rangle} \frac{\sqrt{(\sum_{j=0}^n j^2 c_j^2)(\sum_{j=0}^n j^2 d_j^2)}}{\min_{(s,t) \in \mathbb{T}^2} \|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^2}$$

PROOF By 1.2.1 we have:

$$\begin{aligned}\langle \text{ICN} \rangle &\leq C_{\langle \text{ICN} \rangle} \int_0^{2\pi} \int_0^{2\pi} \frac{\left\| \frac{d\mathbf{r}_1(t)}{dt} \right\| \left\| \frac{d\mathbf{r}_2(s)}{ds} \right\|}{4\pi \min_{(s,t) \in \mathbb{T}^2} \|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^2} dt ds \\ &\leq C_{\langle \text{ICN} \rangle} \int_0^{2\pi} \int_0^{2\pi} \frac{\sqrt{(\sum_{j=0}^n j^2 c_j^2)(\sum_{j=0}^n j^2 d_j^2)}}{4\pi \min_{(s,t) \in \mathbb{T}^2} \|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^2} dt ds \\ &= C_{\langle \text{ICN} \rangle} \pi \frac{\sqrt{(\sum_{j=0}^n j^2 c_j^2)(\sum_{j=0}^n j^2 d_j^2)}}{\min_{(s,t) \in \mathbb{T}^2} \|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^2} dt ds,\end{aligned}$$

so that we may bound $\langle \text{ICN} \rangle$:

$$\langle \text{ICN} \rangle \leq C_{\langle \text{ICN} \rangle} \pi \frac{\sqrt{(\sum_{j=0}^n j^2 c_j^2)(\sum_{j=0}^n j^2 d_j^2)}}{\min_{(s,t) \in \mathbb{T}^2} \|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^2}$$

□

If we make even stronger assumptions, for example that the initial spaces curves be orthogonal, then we can significantly improve the above bound. For example, if the initial space curves $\mathbf{r}_1(t), \mathbf{r}_2(s) : \mathbb{S}^1 \rightarrow \mathbb{R}^{4N+2}$ are orthogonal:

$$\mathbf{r}_1(t) = c_0 \mathbf{e}_1 + \sum_{k=1}^N (c_k \cos(kt) \mathbf{e}_{4k-1} + c_k \sin(kt) \mathbf{e}_{4k}) \quad (2.3.1)$$

$$\mathbf{r}_2(s) = d_0 \mathbf{e}_2 + \sum_{k=1}^N (d_k \cos(ks) \mathbf{e}_{4k+1} + d_k \sin(ks) \mathbf{e}_{4k+2}), \quad (2.3.2)$$

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then by Hadamard's inequality we have that:

$$\langle \text{ICN} \rangle \leq \frac{1}{4\pi} \int_{s=0}^{2\pi} \int_{t=0}^{2\pi} C_{\text{ICN}} \frac{\|\mathbf{r}'_1(t)\| \|\mathbf{r}'_2(s)\|}{\|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^2} ds dt = \pi C_{\text{ICN}} \frac{\sqrt{(\sum_{k=1}^N k^2 c_k^2)(\sum_{k=1}^N k^2 d_k^2)}}{\sum_{k=0}^N c_k^2 + \sum_{k=0}^N d_k^2}$$

Similarly, for the total curvature we have the following theorem:

Theorem 2.3.1. *Let $\mathbf{r}(t) : \mathbb{S}^1 \rightarrow \mathbb{R}^{2n+1}$ be given by:*

$$\begin{aligned} \mathbf{r}(t) &= c_0 \mathbf{e}_0 + \sum_{k=1}^n (c_k \cos(kt) \mathbf{e}_{2k-1} + c_k \sin(kt) \mathbf{e}_{2k}) \\ &= (c_0, c_1 \cos(t), c_1 \sin(t), c_2 \cos(2t), c_2 \sin(2t), \dots, c_n \cos(nt), c_n \sin(nt)), \end{aligned} \quad (2.3.3)$$

then $\langle \kappa(C) \rangle$ is finite and satisfies the following bound:

$$\langle \kappa(C) \rangle \leq 2\pi \sqrt{\frac{\sum_{k=0}^n c_k^2 k^2}{\sum_{k=0}^n c_k^2 k^4}}$$

PROOF For $\langle \kappa(C) \rangle$, notice first that:

$$\begin{aligned} \langle \kappa(C) \rangle &\leq \int_0^{2\pi} \frac{\|\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} dt \\ \|\mathbf{r}'(t)\|^2 &= \sum_{k=1}^n c_k^2 k^2 \\ \|\mathbf{r}''(t)\|^2 &= \sum_{k=1}^n c_k^2 k^4 \end{aligned} \quad (2.3.4)$$

The quotient $\frac{\|\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}$ has no dependence on t , and so the result follows from 2.3.4 by integrating over t . \square

Remark 2.3.2. As mentioned in the introduction, since we are considering arbitrary space curves in \mathbb{R}^n , and since the formulas involved did not have any explicit dependence on n , then we may allow n to approach infinity and consider a space curve of the form:

$$\begin{aligned} \mathbf{r}(t) &= c_0 \mathbf{e}_0 + \sum_{n=1}^{\infty} (c_n \cos(nt) \mathbf{e}_{2n-1} + c_n \sin(nt) \mathbf{e}_{2n}) \\ &= (c_0, c_1 \cos(t), c_1 \sin(t), c_2 \cos(2t), c_2 \sin(2t), \dots, c_n \cos(nt), c_n \sin(nt), \dots) \end{aligned} \quad (2.3.5)$$

In this way, if we define the sequences $\mathbf{c} = \{c_n\}_{n=0}^{\infty}$, $\mathbf{c}' = \{nc_n\}_{n=0}^{\infty}$, $\mathbf{c}'' = \{n^2 c_n\}_{n=0}^{\infty}$ and stipulate that $\|\mathbf{c}''\|_{l^2} < \infty$ (that is, the sequence \mathbf{c} decays faster than $n^{(-5-\epsilon)/2}$), then the quotient $\frac{\|\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}$ will be finite and will give a bound for $\langle \kappa(C) \rangle$ so that with high probability

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the projections will have bounded total curvature given by:

$$\langle \kappa(C) \rangle \leq \int_0^{2\pi} \frac{\|\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} dt = \int_0^{2\pi} \sqrt{\frac{\sum_{n=0}^{\infty} n^4 c_n^2}{\sum_{n=0}^{\infty} n^2 c_n^2}} dt$$

and therefore:

$$\langle \kappa(C) \rangle \leq 2\pi \frac{\|\mathbf{c}''\|_{l^2}}{\|\mathbf{c}'\|_{l^2}}$$

A similar result holds for $\langle \text{ICN} \rangle$. In particular, using the data 2.3.1, we will have that:

$$\langle \text{ICN} \rangle \leq \pi C_{\text{ICN}} \frac{\|\mathbf{c}'\|_{l^2} \|\mathbf{d}'\|_{l^2}}{\|\mathbf{c}\|_{l^2}^2 + \|\mathbf{d}\|_{l^2}^2}, \quad (2.3.6)$$

so that the sequence \mathbf{c} must decay faster than $n^{(-3-\epsilon)/2}$.

Chapter 3

Second Moment of the Linking Number

3.1 Integration Over 3-Dimensional Subspaces

As stated in the introduction, in order to find the second moment of the linking number for a link whose components γ_1 and γ_2 are parametrized by $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$, then we must consider the expectation value of an integral of the form:

$$\text{Lk}^2(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{16\pi^2} \int_{(s,t) \in \mathbb{T}^2} \int_{(s',t') \in \mathbb{T}^2} \frac{\text{Det}[\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t)] \text{Det}[\dot{\mathbf{r}}_1(t'), \dot{\mathbf{r}}_2(s'), \mathbf{r}_2(s') - \mathbf{r}_1(t')]}{\|\mathbf{r}_2(s) - \mathbf{r}_1(t)\|^3 \|\mathbf{r}_2(s') - \mathbf{r}_1(t')\|^3} ds dt ds' dt',$$

So, to begin, define a function $I_{\langle \text{Lk}^2 \rangle} : \text{Mat}_{4,3} \times \text{Mat}_{4,3} \rightarrow \mathbb{R}$ by:

$$I_{\langle \text{Lk}^2 \rangle}(A, A') = \frac{1}{(2\pi)^2} \int_{\mathbf{v} \in \mathbb{R}^4} \frac{\text{Det}([A \frac{\mathbf{v}}{\|\mathbf{v}\|}]) \text{Det}([A' \frac{\mathbf{v}}{\|\mathbf{v}\|}])}{\|\text{proj}_{\mathbf{v}^\perp}(\mathbf{a}_3)\|^3 \|\text{proj}_{\mathbf{v}^\perp}(\mathbf{a}'_3)\|^3} e^{-\|\mathbf{v}\|^2/2} d\mathbf{v}.$$

With this definition we have that:

$$\langle \text{Lk}^2(\mathbf{r}_1, \mathbf{r}_2) \rangle = \frac{1}{(4\pi)^2 (2\pi)^2} \int_{(s,t) \in \mathbb{T}^2} \int_{(s',t') \in \mathbb{T}^2} I_{\langle \text{Lk}^2 \rangle}(\mathbf{R}(s,t), \mathbf{R}(s',t')) ds dt ds' dt'$$

where $\mathbf{R}(s,t) = [\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t)]$ and $\mathbf{R}(s',t') = [\dot{\mathbf{r}}_1(t'), \dot{\mathbf{r}}_2(s'), \mathbf{r}_2(s') - \mathbf{r}_1(t')]$.

Remark 3.1.1. For conciseness, hereinafter we will suppress the dependence on the initial space curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ and will henceforth write $\langle \text{Lk}^2 \rangle$ to denote $\langle \text{Lk}^2(\mathbf{r}_1, \mathbf{r}_2) \rangle$ when the dependence on $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ is clear.

In order to find $\langle \text{Lk}^2 \rangle$, we will need to find an explicit form for the integral $I_{\langle \text{Lk}^2 \rangle}(A, A')$ and then we will need to integrate over the configuration space of pairs of tuples of points on distinct components of the link. We will now focus on performing the integral in the expression for $I_{\langle \text{Lk}^2 \rangle}(A, A')$:

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Lemma 3.1.2. *Given matrices $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$, $A' = [\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3] \in \text{Mat}_{4,3}$ and the function*

$$I_{\langle \text{Lk}^2 \rangle}(A, A') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \frac{\text{Det}([A \frac{\mathbf{v}}{\|\mathbf{v}\|}]) \text{Det}([A' \frac{\mathbf{v}}{\|\mathbf{v}\|}])}{\|\text{proj}_{\mathbf{v}^\perp}(\mathbf{a}_3)\|^3 \|\text{proj}_{\mathbf{v}^\perp}(\mathbf{a}'_3)\|^3} e^{-\|\mathbf{v}\|^2/2} d\mathbf{v}$$

where $[A \frac{\mathbf{v}}{\|\mathbf{v}\|}]$ denotes a new matrix in $\text{Mat}_{4,4}$ with the column $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ appended to the matrix A , then we have

$$\begin{aligned} \frac{\pi}{2} I_{\langle \text{Lk}^2 \rangle}(A, A') = \\ \frac{(\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2) \text{Det}(A^T A') + (\mathbf{a}_3 \cdot \mathbf{a}'_3) \text{Det}([\mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3]) \text{Det}([\mathbf{a}'_3, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3])}{\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 (\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2)^{3/2}} \end{aligned}$$

PROOF To begin, first notice that we may write $I_{\langle \text{Lk}^2 \rangle}(A, A')$ as:

$$I_{\langle \text{Lk}^2 \rangle}(A, A') = \frac{\text{Det}(A^T A)^{1/2} \text{Det}(A'^T A')^{1/2}}{(2\pi)^2 \|\mathbf{a}_3(s, t)\|^3 \|\mathbf{a}'_3(s', t')\|^3} \int_{\mathbb{R}^4} \frac{\text{Det}([Q \frac{\mathbf{v}}{\|\mathbf{v}\|}]) \text{Det}([Q' \frac{\mathbf{v}}{\|\mathbf{v}\|}])}{\|\text{proj}_{\mathbf{v}^\perp}(\mathbf{q}_3)\|^3 \|\text{proj}_{\mathbf{v}^\perp}(\mathbf{q}'_3)\|^3} dV$$

with $Q = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]$ and $Q' = [\mathbf{q}'_1, \mathbf{q}'_2, \mathbf{q}'_3]$ where the vectors \mathbf{q}_i and \mathbf{q}'_i are obtained by applying the Gram-Schmidt algorithm to the ordered set of vectors $\{\mathbf{a}_3, \mathbf{a}_2, \mathbf{a}_1\}$ and $\{\mathbf{a}'_3, \mathbf{a}'_2, \mathbf{a}'_1\}$ respectively.

Now write the matrices Q and Q' in a basis where

$$\mathbf{b}_1 = \frac{\mathbf{q}_3 + \mathbf{q}'_3}{\|\mathbf{q}_3 + \mathbf{q}'_3\|}, \mathbf{b}_2 = \frac{\mathbf{q}_3 - \mathbf{q}'_3}{\|\mathbf{q}_3 - \mathbf{q}'_3\|} \text{ and } \mathbf{b}_3, \mathbf{b}_4 \in \text{span}\{\mathbf{q}_3, \mathbf{q}'_3\}^\perp \quad (3.1.1)$$

Choosing this basis serves to simplify the denominators as much as possible, since they involve norms of projections of the vectors \mathbf{q}_3 and \mathbf{q}'_3 . Now we have that:

$$I_{\langle \text{Lk}^2 \rangle}(Q, Q') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \frac{\text{Det}([Q \frac{\mathbf{v}}{\|\mathbf{v}\|}]) \text{Det}([Q' \frac{\mathbf{v}}{\|\mathbf{v}\|}])}{\|\text{proj}_{\mathbf{v}^\perp}(\mathbf{q}_3)\|^3 \|\text{proj}_{\mathbf{v}^\perp}(\mathbf{q}'_3)\|^3} dV$$

and we may expand $I_{\langle \text{Lk}^2 \rangle}(Q, Q')$ in the basis above to obtain that:

$$\begin{aligned} I_{\langle \text{Lk}^2 \rangle}(Q, Q') = \\ \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \frac{\text{Det} \begin{pmatrix} \mathbf{q}_1 \cdot \mathbf{b}_1 & \mathbf{q}_2 \cdot \mathbf{b}_1 & a & v_1 \\ \mathbf{q}_1 \cdot \mathbf{b}_2 & \mathbf{q}_2 \cdot \mathbf{b}_2 & b & v_2 \\ \mathbf{q}_1 \cdot \mathbf{b}_3 & \mathbf{q}_2 \cdot \mathbf{b}_3 & 0 & v_3 \\ \mathbf{q}_1 \cdot \mathbf{b}_4 & \mathbf{q}_2 \cdot \mathbf{b}_4 & 0 & v_4 \end{pmatrix} \text{Det} \begin{pmatrix} \mathbf{q}'_1 \cdot \mathbf{b}_1 & \mathbf{q}'_2 \cdot \mathbf{b}_1 & a & v_1 \\ \mathbf{q}'_1 \cdot \mathbf{b}_2 & \mathbf{q}'_2 \cdot \mathbf{b}_2 & -b & v_2 \\ \mathbf{q}'_1 \cdot \mathbf{b}_3 & \mathbf{q}'_2 \cdot \mathbf{b}_3 & 0 & v_3 \\ \mathbf{q}'_1 \cdot \mathbf{b}_4 & \mathbf{q}'_2 \cdot \mathbf{b}_4 & 0 & v_4 \end{pmatrix}}{\|\mathbf{v}\|^2 \|\text{proj}_{\mathbf{v}^\perp}(\{a, b, 0, 0\}^T)\|^3 \|\text{proj}_{\mathbf{v}^\perp}(\{a, -b, 0, 0\}^T)\|^3} dV \end{aligned}$$

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where $a = \mathbf{q}_3 \cdot \mathbf{b}_1$ and $b = \mathbf{q}_3 \cdot \mathbf{b}_2$. A computation using the fact that $a^2 + b^2 = 1$ gives

$$\begin{aligned} \|\text{proj}_{\mathbf{v}^\perp}(\{a, b, 0, 0\}^T)\|^3 &= \frac{((av_2 - bv_1)^2 + v_3^2 + v_4^2)^{3/2}}{\|\mathbf{v}\|^3} \text{ and} \\ \|\text{proj}_{\mathbf{v}^\perp}(\{a, -b, 0, 0\}^T)\|^3 &= \frac{((av_2 + bv_1)^2 + v_3^2 + v_4^2)^{3/2}}{\|\mathbf{v}\|^3} \end{aligned}$$

so that $I_{\langle \text{Lk}^2 \rangle}(Q, Q')$ may be rewritten as:

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \frac{\text{Det} \begin{pmatrix} \mathbf{q}_1 \cdot \mathbf{b}_1 & \mathbf{q}_2 \cdot \mathbf{b}_1 & a & v_1 \\ \mathbf{q}_1 \cdot \mathbf{b}_2 & \mathbf{q}_2 \cdot \mathbf{b}_2 & b & v_2 \\ \mathbf{q}_1 \cdot \mathbf{b}_3 & \mathbf{q}_2 \cdot \mathbf{b}_3 & 0 & v_3 \\ \mathbf{q}_1 \cdot \mathbf{b}_4 & \mathbf{q}_2 \cdot \mathbf{b}_4 & 0 & v_4 \end{pmatrix} \text{Det} \begin{pmatrix} \mathbf{q}'_1 \cdot \mathbf{b}_1 & \mathbf{q}'_2 \cdot \mathbf{b}_1 & a & v_1 \\ \mathbf{q}'_1 \cdot \mathbf{b}_2 & \mathbf{q}'_2 \cdot \mathbf{b}_2 & -b & v_2 \\ \mathbf{q}'_1 \cdot \mathbf{b}_3 & \mathbf{q}'_2 \cdot \mathbf{b}_3 & 0 & v_3 \\ \mathbf{q}'_1 \cdot \mathbf{b}_4 & \mathbf{q}'_2 \cdot \mathbf{b}_4 & 0 & v_4 \end{pmatrix}}{\|\mathbf{v}\|^{-4}((av_2 - bv_1)^2 + v_3^2 + v_4^2)^{3/2}((av_2 + bv_1)^2 + v_3^2 + v_4^2)^{3/2}} dV$$

Next expand the product of determinants to obtain a degree 2 polynomial in the v_i . It can be seen that the terms of the resulting degree 2 polynomial involving the products $v_i v_j$ with $i \neq j$ will, by symmetry, not contribute to the integral and therefore we will consider the four relevant integrals:

$$I_{ii} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \frac{\|\mathbf{v}\|^4 v_i^2}{((av_2 - bv_1)^2 + v_3^2 + v_4^2)^{3/2}((av_2 + bv_1)^2 + v_3^2 + v_4^2)^{3/2}} dV,$$

for $i = 1, 2, 3, 4$. For all of these integrals, make the change of coordinates $v'_1 = bv_1$ and $v'_2 = av_2$ so that we have:

$$\begin{aligned} I_{11} &= \frac{1}{(2\pi)^2} \frac{1}{ab^3} \int_{\mathbb{R}^4} \frac{\|\mathbf{v}'\|^4 v_1'^2}{((v_2' - v_1')^2 + v_3^2 + v_4^2)^{3/2}((v_2' + v_1')^2 + v_3^2 + v_4^2)^{3/2}} dV' \\ I_{22} &= \frac{1}{(2\pi)^2} \frac{1}{a^3 b} \int_{\mathbb{R}^4} \frac{\|\mathbf{v}'\|^4 v_2'^2}{((v_2' - v_1')^2 + v_3^2 + v_4^2)^{3/2}((v_2' + v_1')^2 + v_3^2 + v_4^2)^{3/2}} dV' \\ I_{33} = I_{44} &= \frac{1}{(2\pi)^2} \frac{1}{ab} \int_{\mathbb{R}^4} \frac{\|\mathbf{v}'\|^4 v_3^2}{((v_2' - v_1')^2 + v_3^2 + v_4^2)^{3/2}((v_2' + v_1')^2 + v_3^2 + v_4^2)^{3/2}} dV' \end{aligned} \tag{3.1.2}$$

where $\|\mathbf{v}'\|^2 = v_1'^2/b^2 + v_2'^2/a^2 + v_3^2 + v_4^2$. A further change of variables shows that $-b^2 I_{11} + a^2 I_{22} = 0$ and $I_{33} = I_{44}$. We will now focus on the latter two integrals. First we will integrate out the radial dependence by switching to toroidal coordinates (see [17]):

$$\begin{aligned} v_1' &= r \cos(\sigma) \cos(\theta) \\ v_2' &= r \cos(\sigma) \sin(\theta) \\ v_3' &= r \sin(\sigma) \cos(\phi) \\ v_4' &= r \sin(\sigma) \sin(\phi) \end{aligned}$$

where $r \in [0, \infty)$, $\sigma \in [0, \pi/2]$, $\theta \in [0, 2\pi]$, and $\phi \in [0, 2\pi]$, and $dV = r^3 \cos(\sigma) \sin(\sigma) dr d\sigma d\theta d\phi$.

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Doing so reduces the integral to an integral over $\mathbb{S}^3 = Gr(4, 3)$ parametrized by the angles σ , θ , and ϕ . In this coordinate system we have:

$$I_{33} = \frac{1}{(2\pi)^2 ab} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^\infty \frac{k_1^2 \sin^2(\sigma) \cos^2(\phi)}{k_2^{3/2} k_3^{3/2}} \exp\left(-\frac{k_1 r^2}{2}\right) r^3 \cos(\sigma) \sin(\sigma) dV$$

where $k_1 = \sin^2(\sigma) + \cos^2(\sigma)((\cos(\theta)/b)^2 + (\sin(\theta)/a)^2)$
 $k_2 = \cos^2(\sigma)(\sin(\theta) - \cos(\theta))^2 + \sin^2(\sigma)$
 $k_3 = \cos^2(\sigma)(\sin(\theta) + \cos(\theta))^2 + \sin^2(\sigma)$

Now change coordinates by taking $r \rightarrow \frac{r}{\sqrt{k_1}}$ and perform the resulting Gaussian integral along with the integral over the ϕ coordinate to obtain:

$$I_{33} = \frac{1}{2\pi ab} \int_0^{2\pi} \int_0^{\pi/2} \frac{\sin(\sigma)^2}{k_2^{3/2} k_3^{3/2}} \cos(\sigma) \sin(\sigma) d\sigma d\theta$$

Now we know that $k_2^{3/2} k_3^{3/2} = (1 - \cos(\sigma)^4 \sin(2\theta)^2)^{3/2}$ and we further have that:

$$I_{33} = \frac{1}{2\pi ab} \int_0^{2\pi} \int_0^{\pi/2} \frac{\sin(\sigma)^3 \cos(\sigma)}{(1 - \cos(\sigma)^4 \sin(2\theta)^2)^{3/2}} d\sigma d\theta$$

Next make the change of coordinates $\theta \rightarrow \frac{\theta}{2}$ to obtain that:

$$I_{33} = \frac{1}{4\pi ab} \int_0^{\pi/2} \int_0^{4\pi} \frac{\sin(\sigma)^3 \cos(\sigma)}{(1 - \cos(\sigma)^4 \sin(\theta)^2)^{3/2}} d\theta d\sigma =$$

$$\frac{2}{\pi ab} \int_0^{\pi/2} \sin(\sigma)^3 \cos(\sigma) \int_0^{\pi/2} \frac{1}{(1 - \cos(\sigma)^4 \sin(\theta)^2)^{3/2}} d\theta d\sigma$$

Now, recall that the complete elliptic integral of the third kind (see [21]) is defined by:

$$\Pi(n, m) = \int_0^{\pi/2} \frac{1}{(1 - n \sin^2(\sigma)) \sqrt{1 - m \sin^2(\sigma)}} d\sigma \text{ for } 0 < m < 1$$

For the integral over θ , we realize that this is a special case of $\Pi(n, m)$ with $n = m = \cos^4(\sigma)$, namely:

$$\Pi(\cos^4(\sigma), \cos^4(\sigma)) = \int_0^{\pi/2} \frac{1}{(1 - \cos(\sigma)^4 \sin(\theta)^2)^{3/2}} d\theta.$$

Finally, we wish to evaluate the following integral:

$$I_{33}(a, b) = \frac{2}{\pi ab} \int_0^{\pi/2} \sin(\sigma)^3 \cos(\sigma) \Pi(\cos(\sigma)^4, \cos(\sigma)^4) d\sigma$$

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Make the change of coordinates $u = \cos(\sigma)^2$ so the above becomes:

$$I_{33}(a, b) = \frac{1}{\pi ab} \int_0^1 (1 - u) \Pi(u^2, u^2) du$$

A symbolic integration in Mathematica [14] shows that $\int_0^1 (1 - u) \Pi(u^2, u^2) du = 1$ so that:

$$I_{33}(a, b) = I_{44}(a, b) = \frac{1}{\pi ab}$$

Remark 3.1.3. One can compute the above integral by hand by using the identity that $\Pi(u^2, u^2) = E(u^2)/(1 - u^2)$ where $E(u)$ is the complete elliptic integral of the second kind, and using the numerous identities concerning the integration of $E(u)$ along with special values of $E(u)$.

Now expand the product of determinants and compute the relevant integrals to obtain:

$$\begin{aligned} I_{\langle \text{Lk}^2 \rangle}(Q, Q') = & \frac{1}{(2\pi)^2} (I_{11} \text{Det} \begin{pmatrix} \mathbf{q}_1 \cdot \mathbf{b}_2 & \mathbf{q}_2 \cdot \mathbf{b}_2 & b \\ \mathbf{q}_1 \cdot \mathbf{b}_3 & \mathbf{q}_2 \cdot \mathbf{b}_3 & 0 \\ \mathbf{q}_1 \cdot \mathbf{b}_4 & \mathbf{q}_2 \cdot \mathbf{b}_4 & 0 \end{pmatrix} \text{Det} \begin{pmatrix} \mathbf{q}'_1 \cdot \mathbf{b}_2 & \mathbf{q}'_2 \cdot \mathbf{b}_2 & -b \\ \mathbf{q}'_1 \cdot \mathbf{b}_3 & \mathbf{q}'_2 \cdot \mathbf{b}_3 & 0 \\ \mathbf{q}'_1 \cdot \mathbf{b}_4 & \mathbf{q}'_2 \cdot \mathbf{b}_4 & 0 \end{pmatrix} + \\ & I_{22} \text{Det} \begin{pmatrix} \mathbf{q}_1 \cdot \mathbf{b}_2 & \mathbf{q}_2 \cdot \mathbf{b}_2 & a \\ \mathbf{q}_1 \cdot \mathbf{b}_3 & \mathbf{q}_2 \cdot \mathbf{b}_3 & 0 \\ \mathbf{q}_1 \cdot \mathbf{b}_4 & \mathbf{q}_2 \cdot \mathbf{b}_4 & 0 \end{pmatrix} \text{Det} \begin{pmatrix} \mathbf{q}'_1 \cdot \mathbf{b}_2 & \mathbf{q}'_2 \cdot \mathbf{b}_2 & a \\ \mathbf{q}'_1 \cdot \mathbf{b}_3 & \mathbf{q}'_2 \cdot \mathbf{b}_3 & 0 \\ \mathbf{q}'_1 \cdot \mathbf{b}_4 & \mathbf{q}'_2 \cdot \mathbf{b}_4 & 0 \end{pmatrix} + \\ & I_{33} \text{Det} \begin{pmatrix} \mathbf{q}_1 \cdot \mathbf{b}_1 & \mathbf{q}_2 \cdot \mathbf{b}_1 & a \\ \mathbf{q}_1 \cdot \mathbf{b}_2 & \mathbf{q}_2 \cdot \mathbf{b}_2 & b \\ \mathbf{q}_1 \cdot \mathbf{b}_4 & \mathbf{q}_2 \cdot \mathbf{b}_4 & 0 \end{pmatrix} \text{Det} \begin{pmatrix} \mathbf{q}'_1 \cdot \mathbf{b}_1 & \mathbf{q}'_2 \cdot \mathbf{b}_1 & a \\ \mathbf{q}'_1 \cdot \mathbf{b}_2 & \mathbf{q}'_2 \cdot \mathbf{b}_2 & -b \\ \mathbf{q}'_1 \cdot \mathbf{b}_4 & \mathbf{q}'_2 \cdot \mathbf{b}_4 & 0 \end{pmatrix} + \\ & I_{44} \text{Det} \begin{pmatrix} \mathbf{q}_1 \cdot \mathbf{b}_1 & \mathbf{q}_2 \cdot \mathbf{b}_1 & a \\ \mathbf{q}_1 \cdot \mathbf{b}_2 & \mathbf{q}_2 \cdot \mathbf{b}_2 & b \\ \mathbf{q}_1 \cdot \mathbf{b}_3 & \mathbf{q}_2 \cdot \mathbf{b}_3 & 0 \end{pmatrix} \text{Det} \begin{pmatrix} \mathbf{q}'_1 \cdot \mathbf{b}_1 & \mathbf{q}'_2 \cdot \mathbf{b}_1 & a \\ \mathbf{q}'_1 \cdot \mathbf{b}_2 & \mathbf{q}'_2 \cdot \mathbf{b}_2 & -b \\ \mathbf{q}'_1 \cdot \mathbf{b}_3 & \mathbf{q}'_2 \cdot \mathbf{b}_3 & 0 \end{pmatrix}) \end{aligned}$$

Further simplifying and using the fact that $-b^2 I_{11} + a^2 I_{22} = 0$, we find that the I_{11} and I_{22} terms cancel so that we have:

$$\begin{aligned} I_{\langle \text{Lk}^2 \rangle}(Q, Q') = & \frac{1}{\pi ab} (\text{Det} \begin{pmatrix} \mathbf{q}_1 \cdot \mathbf{b}_1 & \mathbf{q}_2 \cdot \mathbf{b}_1 & a \\ \mathbf{q}_1 \cdot \mathbf{b}_2 & \mathbf{q}_2 \cdot \mathbf{b}_2 & b \\ \mathbf{q}_1 \cdot \mathbf{b}_4 & \mathbf{q}_2 \cdot \mathbf{b}_4 & 0 \end{pmatrix} \text{Det} \begin{pmatrix} \mathbf{q}'_1 \cdot \mathbf{b}_1 & \mathbf{q}'_2 \cdot \mathbf{b}_1 & a \\ \mathbf{q}'_1 \cdot \mathbf{b}_2 & \mathbf{q}'_2 \cdot \mathbf{b}_2 & -b \\ \mathbf{q}'_1 \cdot \mathbf{b}_4 & \mathbf{q}'_2 \cdot \mathbf{b}_4 & 0 \end{pmatrix} + \\ & \text{Det} \begin{pmatrix} \mathbf{q}_1 \cdot \mathbf{b}_1 & \mathbf{q}_2 \cdot \mathbf{b}_1 & a \\ \mathbf{q}_1 \cdot \mathbf{b}_2 & \mathbf{q}_2 \cdot \mathbf{b}_2 & b \\ \mathbf{q}_1 \cdot \mathbf{b}_3 & \mathbf{q}_2 \cdot \mathbf{b}_3 & 0 \end{pmatrix} \text{Det} \begin{pmatrix} \mathbf{q}'_1 \cdot \mathbf{b}_1 & \mathbf{q}'_2 \cdot \mathbf{b}_1 & a \\ \mathbf{q}'_1 \cdot \mathbf{b}_2 & \mathbf{q}'_2 \cdot \mathbf{b}_2 & -b \\ \mathbf{q}'_1 \cdot \mathbf{b}_3 & \mathbf{q}'_2 \cdot \mathbf{b}_3 & 0 \end{pmatrix}) \end{aligned}$$

We will now focus on the first term in the previous sum. As in [17], define that:

$$[\mathbf{a} : \mathbf{b} : \mathbf{c}]_i = \sum_{j,k,l=1,\dots,4} \epsilon_{ijkl} a_j b_k c_l \quad (3.1.3)$$

and notice:

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$$\frac{\text{Det}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4, [\mathbf{q}_1 : \mathbf{q}_2 : \mathbf{q}_3]) \text{Det}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{b}_3)}{\mathbf{b}_3 \cdot [\mathbf{q}_1 : \mathbf{q}_2 : \mathbf{q}_3]} = \text{Det} \begin{pmatrix} \mathbf{q}_1 \cdot \mathbf{b}_1 & \mathbf{q}_2 \cdot \mathbf{b}_1 & a \\ \mathbf{q}_1 \cdot \mathbf{b}_2 & \mathbf{q}_2 \cdot \mathbf{b}_2 & b \\ \mathbf{q}_1 \cdot \mathbf{b}_4 & \mathbf{q}_2 \cdot \mathbf{b}_4 & 0 \end{pmatrix}$$

Since

$$\frac{\text{Det}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4, [\mathbf{q}_1 : \mathbf{q}_2 : \mathbf{q}_3])}{\mathbf{b}_3 \cdot [\mathbf{q}_1 : \mathbf{q}_2 : \mathbf{q}_3]} = \frac{[\mathbf{b}_1 : \mathbf{b}_2 : \mathbf{b}_4] \cdot [\mathbf{q}_1 : \mathbf{q}_2 : \mathbf{q}_3]}{\mathbf{b}_3 \cdot [\mathbf{q}_1 : \mathbf{q}_2 : \mathbf{q}_3]} = -1,$$

we have:

$$\text{Det} \begin{pmatrix} \mathbf{q}_1 \cdot \mathbf{b}_1 & \mathbf{q}_2 \cdot \mathbf{b}_1 & a \\ \mathbf{q}_1 \cdot \mathbf{b}_2 & \mathbf{q}_2 \cdot \mathbf{b}_2 & b \\ \mathbf{q}_1 \cdot \mathbf{b}_4 & \mathbf{q}_2 \cdot \mathbf{b}_4 & 0 \end{pmatrix} = -\mathbf{b}_3 \cdot [\mathbf{q}_1 : \mathbf{q}_2 : \mathbf{q}_3] = -\frac{\mathbf{b}_3 \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]}{\|[\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]\|}$$

Remark 3.1.4. Notice that the sign of $[\mathbf{b}_1 : \mathbf{b}_2 : \mathbf{b}_4]$ is not important to this discussion, only the fact that it is a multiple of the omitted basis vector. This will be important in the next chapter.

Similar arguments to the one above applied to the second term of the aforementioned sum gives:

$$I_{\langle \text{Lk}^2 \rangle}(Q, Q') = \frac{1}{\pi ab} \left(\frac{\mathbf{b}_3 \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]}{\|[\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]\|} \frac{\mathbf{b}_3 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3]}{\|[\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3]\|} + \frac{\mathbf{b}_4 \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]}{\|[\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]\|} \frac{\mathbf{b}_4 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3]}{\|[\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3]\|} \right)$$

This way we have:

$$I_{\langle \text{Lk}^2 \rangle}(A, A') = \frac{\mathbf{b}_3 \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3] \mathbf{b}_3 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3] + \mathbf{b}_4 \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3] \mathbf{b}_4 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3]}{\pi ab \|\mathbf{a}_3\|^3 \|\mathbf{a}'_3\|^3} \quad (3.1.4)$$

The above may be simplified further using the fact that:

$$ab = \frac{(\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2)^{1/2}}{2\|\mathbf{a}_3\| \|\mathbf{a}'_3\|} \quad (3.1.5)$$

For the preceding argument, \mathbf{b}_3 and \mathbf{b}_4 were simply chosen to be orthogonal to the span of \mathbf{a}_3 and \mathbf{a}'_3 however particular choices will result in further simplified results. For example, choosing

$$\mathbf{b}_3 = \frac{[\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]]}{\|[\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]]\|}$$

$$\mathbf{b}_4 = \frac{[\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]]]}{\|[\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]]]\|},$$

3.1. Integration Over 3-Dimensional Subspaces

will give that $\mathbf{b}_3 \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3] = 0$ so that all remains to compute is $\mathbf{b}_4 \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3] \mathbf{b}_4 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3]$. By writing $\widetilde{\mathbf{b}}_4 = [\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]]]$, we have:

$$\begin{aligned} \mathbf{b}_4 \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3] \mathbf{b}_4 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3] &= \\ &= \frac{d_1(A, A') d_2(A, A')}{\|\widetilde{\mathbf{b}}_4\|^2} \end{aligned}$$

where

$$\begin{aligned} d_1(A, A') &= \text{Det}([\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3], \mathbf{a}_3, \mathbf{a}'_3, [\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]]) \\ d_2(A, A') &= \text{Det}([\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3], \mathbf{a}_3, \mathbf{a}'_3, [\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]]) \end{aligned}$$

We will compute $d_1(A, A') d_2(A, A')$ by computing the determinant of the matrix:

$$\begin{pmatrix} [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3] \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3] & 0 & [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3] \cdot \mathbf{a}'_3 & 0 \\ \mathbf{a}_3 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3] & \|\mathbf{a}_3\|^2 & \mathbf{a}_3 \cdot \mathbf{a}'_3 & 0 \\ 0 & \mathbf{a}'_3 \cdot \mathbf{a}_3 & \|\mathbf{a}'_3\|^2 & 0 \\ [\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]] \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3] & 0 & 0 & \|[\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]]\|^2 \end{pmatrix}$$

Computing the determinant of the above matrix and diving by $\|\widetilde{\mathbf{b}}_4\|^2$ we obtain:

$$\begin{aligned} \frac{d_1(A, A') d_2(A, A')}{\|\widetilde{\mathbf{b}}_4\|^2} &= \\ \frac{\|[\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]]\|^2}{\|\widetilde{\mathbf{b}}_4\|^2} \text{Det} \begin{pmatrix} [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3] \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3] & 0 & [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3] \cdot \mathbf{a}'_3 \\ \mathbf{a}_3 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3] & \|\mathbf{a}_3\|^2 & \mathbf{a}_3 \cdot \mathbf{a}'_3 \\ 0 & \mathbf{a}'_3 \cdot \mathbf{a}_3 & \|\mathbf{a}'_3\|^2 \end{pmatrix} \end{aligned}$$

Now we may use the fact that (see [17] for example):

$$\|[\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]]]\|^2 = \|[\mathbf{a}_3 : \mathbf{a}'_3 : [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3]]\|^2 (\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2)$$

to write (3.1.4) as:

$$\begin{aligned} I_{\langle \text{Lk}^2 \rangle}(A, A') &= \\ \frac{(\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2) \text{Det}(A^T A') + (\mathbf{a}_3 \cdot \mathbf{a}'_3)(\mathbf{a}_3 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3])(\mathbf{a}_3 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3])}{\pi ab \|\mathbf{a}_3\|^3 \|\mathbf{a}'_3\|^3 (\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2)} \end{aligned}$$

which may be further simplified using (3.1.5):

$$\begin{aligned} I_{\langle \text{Lk}^2 \rangle}(A, A') &= \\ 2 \frac{(\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2) \text{Det}(A^T A') + (\mathbf{a}_3 \cdot \mathbf{a}'_3)(\mathbf{a}_3 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3])(\mathbf{a}'_3 \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3])}{\pi \|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 (\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2)^{3/2}} \end{aligned} \tag{3.1.6}$$

3.2. Integration Over the Stiefel Manifold

Finally, we may write the above as a function of the inner products of the initial data $\{\mathbf{a}_i, \mathbf{a}'_i\}_{i=1}^3$ since

$$\text{Det}(A^T A') = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \cdot [\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3] = \text{Det} \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{a}'_1 & \mathbf{a}_1 \cdot \mathbf{a}'_2 & \mathbf{a}_1 \cdot \mathbf{a}'_3 \\ \mathbf{a}_2 \cdot \mathbf{a}'_1 & \mathbf{a}_2 \cdot \mathbf{a}'_2 & \mathbf{a}_2 \cdot \mathbf{a}'_3 \\ \mathbf{a}_3 \cdot \mathbf{a}'_1 & \mathbf{a}_3 \cdot \mathbf{a}'_2 & \mathbf{a}_3 \cdot \mathbf{a}'_3 \end{pmatrix}$$

and

$$(\mathbf{a}_3 \cdot [\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3])(\mathbf{a}'_3 \cdot [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]) = \text{Det} \begin{pmatrix} \mathbf{a}_3 \cdot \mathbf{a}'_3 & \mathbf{a}_3 \cdot \mathbf{a}_1 & \mathbf{a}_3 \cdot \mathbf{a}_2 & \mathbf{a}_3 \cdot \mathbf{a}_3 \\ \mathbf{a}'_1 \cdot \mathbf{a}'_3 & \mathbf{a}'_1 \cdot \mathbf{a}_1 & \mathbf{a}'_1 \cdot \mathbf{a}_2 & \mathbf{a}'_1 \cdot \mathbf{a}_3 \\ \mathbf{a}'_2 \cdot \mathbf{a}'_3 & \mathbf{a}'_2 \cdot \mathbf{a}_1 & \mathbf{a}'_2 \cdot \mathbf{a}_2 & \mathbf{a}'_2 \cdot \mathbf{a}_3 \\ \mathbf{a}'_3 \cdot \mathbf{a}'_3 & \mathbf{a}'_3 \cdot \mathbf{a}_1 & \mathbf{a}'_3 \cdot \mathbf{a}_2 & \mathbf{a}'_3 \cdot \mathbf{a}_3 \end{pmatrix}$$

□

Remark 3.1.5. Notice that the integrals I_{11} and I_{22} diverge, however the change of coordinates made in the course of the proof resulted in these terms exactly canceling out.

3.2 Integration Over the Stiefel Manifold

In the previous section we averaged the value of Lk^2 over all the orthogonal projections of a pair of space curves in \mathbb{R}^4 to 3 dimensional spaces, so that in fact we found the average of Lk^2 over $Gr(4, 3)$, the Grassmannian of 3 dimensional subspaces in \mathbb{R}^4 . Notice that (3.1.6) is a function of the inner products of the initial data, and has nothing in it that depends crucially on the assumption that our original space curves be embedded in \mathbb{R}^4 , and we therefore imagine that if $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3 \in \mathbb{R}^n$ and $\dim(\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3\}) = 4$, then the result (3.1.6) should hold when averaging over $Gr(n, 3)$ with its unique normalized $O(n)$ invariant measure. The main theorem of this paper is the following:

Theorem 1.2.3. *Given two closed, differentiable, non-intersecting space curves $\gamma_1, \gamma_2 : \mathbb{S}^1 \rightarrow \mathbb{R}^n$ with parametrizations $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$, and let $\text{proj}(\gamma_1), \text{proj}(\gamma_2)$ denote the orthogonal projections of γ_1 and γ_2 on to the 3 dimensional subspace spanned by the columns of U , then the expected value of $\text{Lk}^2(\text{proj}(\gamma_1), \text{proj}(\gamma_2))$, denoted $\langle \text{Lk}^2 \rangle$, averaged over all orthogonal projections to 3 dimensional subspaces with respect to the unique normalized $O(n)$ -invariant measure on $Gr(n, 3)$, is given by the following integral:*

$$\langle \text{Lk}^2 \rangle = \frac{1}{16\pi^2} \int_{(s,t) \in \mathbb{T}^2} \int_{(s',t') \in \mathbb{T}^2} I_{\langle \text{Lk}^2 \rangle}(A(s, t), A'(s', t')) ds dt ds' dt' \quad (3.2.1)$$

where

$$\begin{aligned} & \frac{\pi}{2} I_{\langle \text{Lk}^2 \rangle}(A(s, t), A'(s', t')) = \\ & \frac{(\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2) \text{Det}(A^T A') + (\mathbf{a}_3 \cdot \mathbf{a}'_3) \text{Det}([\mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3]) \text{Det}([\mathbf{a}'_3, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3])}{\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 (\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 - (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2)^{3/2}} \end{aligned}$$

3.2. Integration Over the Stiefel Manifold

and

$$\begin{aligned}\mathbf{a}_1 &= \dot{\mathbf{r}}_1(t), \mathbf{a}_2 = \dot{\mathbf{r}}_2(s), \mathbf{a}_3 = \mathbf{r}_2(s) - \mathbf{r}_1(t) \\ \mathbf{a}'_1 &= \dot{\mathbf{r}}_1(t'), \mathbf{a}'_2 = \dot{\mathbf{r}}_2(s'), \mathbf{a}'_3 = \mathbf{r}_2(s') - \mathbf{r}_1(t') \\ A &= [\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t)] \\ A' &= [\dot{\mathbf{r}}_1(t'), \dot{\mathbf{r}}_2(s'), \mathbf{r}_2(s') - \mathbf{r}_1(t')]\end{aligned}$$

To show this, we first need to introduce the Stiefel manifold $V_{n,m}$ of orthonormal m -frames in \mathbb{R}^n (in what follows, we will use the normalization conventions in [13] and [22]):

Definition 3.2.1. For $n \geq m$, the Stiefel manifold, $V_{n,m}$ of orthonormal m -frames in \mathbb{R}^n is the set of matrices $M \in \text{Mat}_{n,m}$ such that $M^T M = \mathbf{1}_m$.

There is a projection map, π , from $V_{n,m}$ to $Gr(n, m)$ where the set of vectors, M , is mapped to $\text{span}\{M\}$, and moreover any right $O(m)$ invariant function $f(x)$ on $V_{n,m}$ gives a function $F(\pi(x)) = f(x)$ on $Gr(n, m)$. Now we will fix the unique normalized $O(n)$ invariant measures dv on $V_{n,k}$ and dy on $Gr(n, k)$, the former which is normalized by:

$$\sigma_{n,k} = \int_{V_{n,k}} dv = \frac{\pi^{kn/2}}{\Gamma_k(n/2)}$$

where $\Gamma_k(n/2) = (2\pi)^{(k^2-k)/4} \prod_{i=1}^k \Gamma(n/2 - \frac{1}{2}(i-1))$ and $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the gamma function, and the latter, dy , which is normalized by stipulating that:

$$\int_{V_{n,m}} f(x) dx = \int_{Gr(n,m)} F(y) dy. \quad (3.2.2)$$

We will now quote a result concerning a decomposition of the measure dx on $\text{Mat}_{n,k}$ (induced by the polar decomposition of the matrix x), which is proved in [13] and [22]. As a set up, identify $\Omega_{k,k}$ with the set of symmetric positive-definite k by k matrices viewed as a subspace of $\mathbb{R}^{\binom{k+1}{2}}$. With this identification we have a measure on $\Omega_{k,k}$ given by $d\tilde{r}(r) = \text{Det}(r)^{-(k+1)/2} dr$ where dr is the Lebesgue measure. Now we will invoke the real version of lemma 3.1 in G. Zhang's paper:[22].

Lemma 3.1 in [22]: Let dx , dv and $d\tilde{r}(r)$ be the normalized measures on $\text{Mat}_{n,k}$, $V_{n,k}$ and $\Omega_{k,k}$ defined above. Almost every $x \in \text{Mat}_{n,k}$ may be decomposed as:

$$x = vr^{1/2}$$

where $x \in V_{n,k}$ and $r \in \Omega_{k,k}$. Under this decomposition the measure dx is given by:

$$dx = C_0 \text{Det}(r)^{n/2} dv d\tilde{r}(r) = C_0 \text{Det}(r)^{(n-k-1)/2} dv dr,$$

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namely,

$$\int_{\text{Mat}_{n,k}} f(x) dx = C_0 \int_{V_{n,k}} \int_{\Omega_{k,k}} f(vr^{1/2}) dv dr,$$

where

$$C_0 = \frac{\pi^{nk/2}}{\Gamma_k(n/2)}$$

If the function $f(vr^{1/2})$ is invariant under multiplying by $(r^{1/2})^T$ and if we weight the integrand of the integral over $\text{Mat}_{n,k}$ by a factor of the form $\frac{1}{\pi^{nk/2}} \exp(-x^T x)$ (so, we are using the Gaussian measure) then the decomposition of the integral above becomes:

$$\frac{1}{\pi^{nk/2}} \int_{\text{Mat}_{n,k}} f(x) \exp(-x^T x) dx = \int_{V_{n,k}} f(v) dv \quad (3.2.3)$$

Given the normalization of the measure dy on the Grassmannian given above, then using 3.2.3 we see that we may compute the integral over the Grassmannian by integrating over $\text{Mat}_{n,3}$ using the Gaussian measure, where certain symmetries of the integral may become more apparent. With this set up, we are now ready to prove Theorem 1.2.3.

PROOF First define the function:

$$I_{\langle \text{Lk}^2 \rangle} : \text{Mat}_{n,3} \times \text{Mat}_{n,3} \rightarrow \mathbb{R}$$

by the integral:

$$I_{\langle \text{Lk}^2 \rangle}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], [\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3]) = \int_{U \in \text{Mat}_{n,3}} \frac{\text{Det}(U^T [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]) \text{Det}(U^T [\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3])}{\|U^T \mathbf{a}_3\|^3 \|U^T \mathbf{a}'_3\|^3} dU,$$

where we think of $U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ as an n by 3 matrix comprised of the column vectors $\mathbf{u}_1, \mathbf{u}_2$, and \mathbf{u}_3 and $dU = \frac{1}{(\pi)^{3n/2}} \exp(-\text{Tr}(U^T U)) d\mathbf{u}_1 d\mathbf{u}_2 d\mathbf{u}_3$. Since the above Gaussian measure is invariant under the action of the orthogonal group, then we may assume that:

$$\text{span}(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3\}) \subseteq \text{span}(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}),$$

and moreover we can take $\text{span}(\{\mathbf{a}_3, \mathbf{a}'_3\}) \subseteq \text{span}(\{\mathbf{e}_1, \mathbf{e}_2\})$ where the set $\{\mathbf{e}_i\}_{i=1}^6$ is the collection of the first 6 standard basis vectors in \mathbb{R}^n . Next, choose the basis

$$\begin{aligned} \mathbf{b}_1 &= \frac{\mathbf{a}_3 \|\mathbf{a}'_3\| + \mathbf{a}'_3 \|\mathbf{a}_3\|}{\|\mathbf{a}_3\| \|\mathbf{a}'_3\| + \mathbf{a}'_3 \|\mathbf{a}_3\|}, \mathbf{b}_2 = \frac{\mathbf{a}_3 \|\mathbf{a}'_3\| - \mathbf{a}'_3 \|\mathbf{a}_3\|}{\|\mathbf{a}_3\| \|\mathbf{a}'_3\| - \mathbf{a}'_3 \|\mathbf{a}_3\|} \\ \mathbf{b}_i &= \mathbf{e}_i \text{ for } i \in \{3, 4, \dots, n\} \end{aligned}$$

and express

$$I_{\langle \text{Lk}^2 \rangle}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], [\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3]) = \frac{I_{\langle \text{Lk}^2 \rangle}([\mathbf{a}_1, \mathbf{a}_2, \frac{\mathbf{a}_3}{\|\mathbf{a}_3\|}], [\mathbf{a}'_1, \mathbf{a}'_2, \frac{\mathbf{a}'_3}{\|\mathbf{a}'_3\|}])}{\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2}$$

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in the basis $\{\mathbf{b}_i\}_{i=1}^n$. Next, using the multilinearity of the numerator of the integrand, we may write that:

$$I_{\langle \text{Lk}^2 \rangle}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], [\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3]) = \frac{\sum_{i,j,k,l=1}^6 a_{1i} a_{2j} a'_{1k} a'_{2l} I_{\langle \text{Lk}^2 \rangle}([\mathbf{e}_i, \mathbf{e}_j, a\mathbf{e}_1 + b\mathbf{e}_2], [\mathbf{e}_k, \mathbf{e}_l, a\mathbf{e}_1 - b\mathbf{e}_2])}{\|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2},$$

where $a_{1i}, a_{2j}, a'_{1k}, a'_{2l}$ are the coordinates of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}'_1$ and \mathbf{a}'_2 with the respect to the chosen basis, and

$$a = \frac{\mathbf{a}_3}{\|\mathbf{a}_3\|} \cdot \mathbf{b}_1, \quad b = \frac{\mathbf{a}'_3}{\|\mathbf{a}'_3\|} \cdot \mathbf{b}_2.$$

Next, integrate over all but the first 6 coordinates of the vectors $\mathbf{u}_1, \mathbf{u}_2$, and \mathbf{u}_3 to obtain that:

$$\begin{aligned} & I_{\langle \text{Lk}^2 \rangle}([\mathbf{e}_i, \mathbf{e}_j, a\mathbf{e}_1 + b\mathbf{e}_2], [\mathbf{e}_k, \mathbf{e}_l, a\mathbf{e}_1 - b\mathbf{e}_2]) \\ &= \int_{U \in \text{Mat}_{6,3}} \frac{\text{Det}(U^T [\mathbf{e}_i, \mathbf{e}_j, a\mathbf{e}_1 + b\mathbf{e}_2]) \text{Det}(U^T [\mathbf{e}_k, \mathbf{e}_l, a\mathbf{e}_1 - b\mathbf{e}_2])}{\|U^T(a\mathbf{e}_1 + b\mathbf{e}_2)\|^3 \|U^T(a\mathbf{e}_1 - b\mathbf{e}_2)\|^3} dU, \end{aligned}$$

where now $dU = \frac{1}{(\pi)^{3 \cdot 6/2}} \exp(-\text{Tr}(U^T U)) d\mathbf{u}_1 d\mathbf{u}_2 d\mathbf{u}_3$. Written in the form above, it is then clear that the integral, $I_{\langle \text{Lk}^2 \rangle}([\mathbf{e}_i, \mathbf{e}_j, a\mathbf{e}_1 + b\mathbf{e}_2], [\mathbf{e}_k, \mathbf{e}_l, a\mathbf{e}_1 - b\mathbf{e}_2])$, is nonzero only when $\dim(\text{span}\{\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l, a\mathbf{e}_1 - b\mathbf{e}_2, a\mathbf{e}_1 + b\mathbf{e}_2\}) \leq 4$. Given this observation and further using the invariance of the integral under the orothogonal group, it suffices to considered only when:

$$\text{span}\{\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l, a\mathbf{e}_1 - b\mathbf{e}_2, a\mathbf{e}_1 + b\mathbf{e}_2\} \subseteq \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}.$$

In this way, from now on we assume that $i, j, k, l \in \{1, 2, 3, 4\}$, and moreover that we have integrated over all but the first 4 components of the vectors $\mathbf{u}_1, \mathbf{u}_2$ and \mathbf{u}_3 so that:

$$\begin{aligned} & I_{\langle \text{Lk}^2 \rangle}([\mathbf{e}_i, \mathbf{e}_j, a\mathbf{e}_1 + b\mathbf{e}_2], [\mathbf{e}_k, \mathbf{e}_l, a\mathbf{e}_1 - b\mathbf{e}_2]) \\ &= \int_{U \in \text{Mat}_{4,3}} \frac{\text{Det}(U^T [\mathbf{e}_i, \mathbf{e}_j, a\mathbf{e}_1 + b\mathbf{e}_2]) \text{Det}(U^T [\mathbf{e}_k, \mathbf{e}_l, a\mathbf{e}_1 - b\mathbf{e}_2])}{\|U^T(a\mathbf{e}_1 + b\mathbf{e}_2)\|^3 \|U^T(a\mathbf{e}_1 - b\mathbf{e}_2)\|^3} dU, \end{aligned}$$

where $dU = \frac{1}{(\pi)^{3 \cdot 4/2}} \exp(-\text{Tr}(U^T U)) d\mathbf{u}_1 d\mathbf{u}_2 d\mathbf{u}_3$. Lastly, using lemma 3.1 in [22] we have that:

$$\begin{aligned} & I_{\langle \text{Lk}^2 \rangle}([\mathbf{e}_i, \mathbf{e}_j, a\mathbf{e}_1 + b\mathbf{e}_2], [\mathbf{e}_k, \mathbf{e}_l, a\mathbf{e}_1 - b\mathbf{e}_2]) \\ &= \int_{v \in \mathbb{V}_{4,3}} \frac{\text{Det}(v^T [\mathbf{e}_i, \mathbf{e}_j, a\mathbf{e}_1 + b\mathbf{e}_2]) \text{Det}(v^T [\mathbf{e}_k, \mathbf{e}_l, a\mathbf{e}_1 - b\mathbf{e}_2])}{\|v^T(a\mathbf{e}_1 + b\mathbf{e}_2)\|^3 \|v^T(a\mathbf{e}_1 - b\mathbf{e}_2)\|^3} dv \\ &= \int_{y \in \text{Gr}_{4,3}} \frac{\text{Det}(y^T [\mathbf{e}_i, \mathbf{e}_j, a\mathbf{e}_1 + b\mathbf{e}_2]) \text{Det}(y^T [\mathbf{e}_k, \mathbf{e}_l, a\mathbf{e}_1 - b\mathbf{e}_2])}{\|y^T(a\mathbf{e}_1 + b\mathbf{e}_2)\|^3 \|y^T(a\mathbf{e}_1 - b\mathbf{e}_2)\|^3} dy \\ &= \int_{y' \in \text{Gr}_{4,1}} \frac{\text{Det}([\mathbf{e}_i, \mathbf{e}_j, a\mathbf{e}_1 + b\mathbf{e}_2, y']) \text{Det}([\mathbf{e}_k, \mathbf{e}_l, a\mathbf{e}_1 - b\mathbf{e}_2, y'])}{\|\text{proj}_{y'^\perp}(a\mathbf{e}_1 + b\mathbf{e}_2)\|^3 \|\text{proj}_{y'^\perp}(a\mathbf{e}_1 - b\mathbf{e}_2)\|^3} dy', \end{aligned}$$

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where y' is obtained from the isomorphism between $Gr_{4,3}$ and $Gr_{4,1}$. The result now follows by applying lemma 3.1.2. \square

Remark 3.2.2. Notice that in the previous proof when we applied lemma 3.1 in [22], that we easily integrated over the $\Omega_{3,3}$ factor by using the fact that the linking number is invariant under ambient homeomorphisms of \mathbb{R}^3 , and that multiplying by $(r^{1/2})^T$ induces such a homeomorphism.

Remark 3.2.3. If we write that $I_{ijkl}(a, b) = I_{\langle \text{Lk}^2 \rangle}([e_i, e_j, ae_1 + be_2], [e_k, e_l, ae_1 - be_2])$, then we may list all values of $I_{ijkl(a,b)}$ appearing in the previous proof:

$$I_{ijkl}(a, b) = \begin{cases} \frac{2b^2}{\pi\sqrt{1-(a^2-b^2)^2}} & \text{if } j = k, i = l, i \neq j \text{ and } i, j \in \{1, 3, 4, 5, 6\} \\ \frac{-2b^2}{\pi\sqrt{1-(a^2-b^2)^2}} & \text{if } i = k, j = l, i \neq j \text{ and } i, j \in \{1, 3, 4, 5, 6\} \\ \frac{-2a^2}{\pi\sqrt{1-(a^2-b^2)^2}} & \text{if } j = k, i = l, i \neq j \text{ and } i, j \in \{2, 3, 4, 5, 6\} \\ \frac{2a^2}{\pi\sqrt{1-(a^2-b^2)^2}} & \text{if } i = k, j = l, i \neq j \text{ and } i, j \in \{2, 3, 4, 5, 6\} \\ \frac{2ab}{\pi\sqrt{1-(a^2-b^2)^2}} & \text{if } l = 1, j = 2, i = k, i \in \{3, 4, 5, 6\} \text{ or} \\ & i = 1, l = 2, j = k, j \in \{3, 4, 5, 6\} \text{ or} \\ & k = 1, i = 2, j = l, j \in \{3, 4, 5, 6\} \text{ or} \\ & j = 1, k = 2, i = l, i \in \{3, 4, 5, 6\} \\ \frac{-2ab}{\pi\sqrt{1-(a^2-b^2)^2}} & \text{if } j = 1, l = 2, i = k, i \in \{3, 4, 5, 6\} \text{ or} \\ & l = 1, i = 2, j = k, j \in \{3, 4, 5, 6\} \text{ or} \\ & i = 1, k = 2, j = l, j \in \{3, 4, 5, 6\} \text{ or} \\ & k = 1, j = 2, i = l, i \in \{3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases}$$

Remark 3.2.4. A numerical method very similar to this one is presented in the appendix, and may provide a way to obtain all moments of the linking number.

3.3 Integration Over the Configuration Space

Using (3.1.4) we may find some conditions on the initial pair of spaces curves so as to bound the the second moment of the linking number. As in Theorem (1.2.3), set:

$$\begin{aligned} \mathbf{a}_1(t) &= \dot{\mathbf{r}}_1(t), \mathbf{a}_2(s) = \dot{\mathbf{r}}_2(s), \mathbf{a}_3(s, t) = \mathbf{r}_2(s) - \mathbf{r}_1(t) \\ \mathbf{a}'_1(t') &= \dot{\mathbf{r}}_1(t'), \mathbf{a}'_2(s') = \dot{\mathbf{r}}_2(s'), \mathbf{a}'_3(s', t') = \mathbf{r}_2(s') - \mathbf{r}_1(t') \\ A(s, t) &= [\dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(s), \mathbf{r}_2(s) - \mathbf{r}_1(t)] \\ A'(s', t') &= [\dot{\mathbf{r}}_1(t'), \dot{\mathbf{r}}_2(s'), \mathbf{r}_2(s') - \mathbf{r}_1(t')] \end{aligned}$$

Theorem 1.2.4. *With the definitions above, let $v_1 = \max_{t \in \mathbb{S}^1} \|\dot{\mathbf{r}}_1(t)\|$, $v_2 = \max_{s \in \mathbb{S}^1} \|\dot{\mathbf{r}}_2(s)\|$,*

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$k = \min_{(s,t) \in \mathbb{T}^2} \|\mathbf{r}_2(t) - \mathbf{r}_1(s)\|$ and

$$C = \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{\|\mathbf{a}_3(s,t)\|^2 \|\mathbf{a}'_3(s',t')\|^2 - (\mathbf{a}_3(s,t) \cdot \mathbf{a}'_3(s',t'))^2}} ds dt ds' dt',$$

then

$$\langle \text{Lk}^2 \rangle \leq \frac{1}{(4\pi)^2} \frac{4C v_1^2 v_2^2}{\pi k^2}$$

PROOF The relevant configuration space to integrate over will be the set of pairs of points, one pair per component of the link. Since

$$\langle \text{Lk}^2 \rangle = \frac{1}{(4\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} I_{\langle \text{Lk}^2 \rangle}(A(s,t), A'(s',t')) ds dt ds' dt'$$

then we may find an upper bound by first bounding $I_{\langle \text{Lk}^2 \rangle}(A(s,t), A'(s',t'))$ and then computing the integral over the configuration space. To do so, recall from 3.1.4 that:

$$I_{\langle \text{Lk}^2 \rangle}(A, A') = \frac{\mathbf{b}_3 \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3] \mathbf{b}_3 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3] + \mathbf{b}_4 \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \mathbf{a}_3] \mathbf{b}_4 \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \mathbf{a}'_3]}{\pi ab \|\mathbf{a}_3\|^3 \|\mathbf{a}'_3\|^3}$$

so that we have:

$$I_{\langle \text{Lk}^2 \rangle}(A(s,t), A'(s',t')) \leq \frac{4 \|\mathbf{a}_1(t)\| \|\mathbf{a}'_1(t')\| \|\mathbf{a}_2(s)\| \|\mathbf{a}'_2(s')\|}{\pi \|\mathbf{a}_3(s,t)\| \|\mathbf{a}'_3(s',t')\| \sqrt{\|\mathbf{a}_3(s,t)\|^2 \|\mathbf{a}'_3(s',t')\|^2 - (\mathbf{a}_3(s,t) \cdot \mathbf{a}'_3(s',t'))^2}}, \quad (3.3.1)$$

by Hadamard's inequality. Similar to the $\langle \text{ICN} \rangle$ case, we also have a parameter keeping track of the minimum distance between the two components of the link:

$$k = \min_{(s,t) \in \mathbb{T}^2} \|\mathbf{r}_2(t) - \mathbf{r}_1(s)\| > 0 \quad (3.3.2)$$

Define three more parameters C , v_1 and v_2 , the first of which accounts for the integration over the configuration space, and the remaining two which bound the speed of the components of the link:

$$C = \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{\|\mathbf{a}_3(s,t)\|^2 \|\mathbf{a}'_3(s',t')\|^2 - (\mathbf{a}_3(s,t) \cdot \mathbf{a}'_3(s',t'))^2}} ds dt ds' dt'$$

$$v_1 = \max_{s \in \mathbb{T}} \|\mathbf{a}_1(s)\|$$

$$v_2 = \max_{t \in \mathbb{T}} \|\mathbf{a}_2(t)\|$$

In this way we obtain:

$$\langle \text{Lk}^2 \rangle \leq \frac{1}{(4\pi)^2} \frac{4C v_1^2 v_2^2}{\pi k^2}$$

□

The integrand in the definition of C has a singularity whenever $\mathbf{a}_3(s,t)$ and $\mathbf{a}'_3(s',t')$ are parallel or anti-parallel, and we will now demonstrate how to stipulate further conditions on

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the input data so that C is finite. To this end, split the integral in to two parts, one close to the diagonal where $s = s'$ and $t = t'$ (which we will denote $\mu_1(\epsilon)$) and another away from the diagonal (denoted $\mu_1(\epsilon)^c$), by defining

$$\mu_1(\epsilon) = \{(s', t', s, t) \in \mathbb{T}^4 \text{ where } |s - s'| < \epsilon \text{ and } |t - t'| < \epsilon\}$$

and writing $C = C_1 + C_2$ where:

$$C_1 = \int_{s=0}^{2\pi} \int_{t=0}^{2\pi} \int_{|s-s'| < \epsilon} \int_{|t-t'| < \epsilon} \frac{1}{\sqrt{\|\mathbf{a}_3(s, t)\|^2 \|\mathbf{a}'_3(s', t')\|^2 - (\mathbf{a}_3(s, t) \cdot \mathbf{a}'_3(s', t'))^2}} ds' dt' ds dt$$

$$C_2 = \int_{(s, t, s', t') \in \mu_1^c(\epsilon)} \frac{1}{\sqrt{\|\mathbf{a}_3(s, t)\|^2 \|\mathbf{a}'_3(s', t')\|^2 - (\mathbf{a}_3(s, t) \cdot \mathbf{a}'_3(s', t'))^2}} ds' dt' ds dt > 0$$

Next define that

$$\eta_1(\epsilon) = \min_{(s, t, s', t') \in \mu_1} \sqrt{\frac{\|\mathbf{a}_3(s, t)\|^2 \|\mathbf{a}'_3(s', t')\|^2 - (\mathbf{a}_3(s, t) \cdot \mathbf{a}'_3(s', t'))^2}{(s - s')^2 + (t - t')^2}}$$

$$\eta_2(\epsilon) = \min_{(s, t, s', t') \in \mu_1^c} \sqrt{\|\mathbf{a}_3(s, t)\|^2 \|\mathbf{a}'_3(s', t')\|^2 - (\mathbf{a}_3(s, t) \cdot \mathbf{a}'_3(s', t'))^2}$$

When $\eta_1(\epsilon) > 0$, then we may bound C_1 as:

$$C_1 \leq \int_{s=0}^{2\pi} \int_{t=0}^{2\pi} \int_{|s-s'| < \epsilon} \int_{|t-t'| < \epsilon} \frac{1}{\eta_1(\epsilon)} \frac{1}{\sqrt{(s - s')^2 + (t - t')^2}} ds' dt' ds dt$$

Finally, when we stipulate that v_1 and v_2 are finite, then we have a more precise bound on $\langle \text{Lk}^2 \rangle$:

$$\langle \text{Lk}^2 \rangle \leq \min_{\epsilon} \frac{1}{(4\pi)^2} \frac{4v_1^2 v_2^2}{\pi k^2} \left(C_1 + \frac{\text{vol}(\mu_1^c(\epsilon))}{\eta_2(\epsilon)} \right) \quad (3.3.3)$$

Remark 3.3.1. If we further restrict the data to be as in 2.3.1, then we can get an even more explicit bound on $\langle \text{Lk}^2 \rangle$. That is, starting from 3.3.1, we have:

$$\langle \text{Lk}^2 \rangle \leq \frac{4}{\pi} \frac{(\sum_{k=1}^N k^2 c_k^2)(\sum_{k=1}^N k^2 d_k^2)}{\sum_{k=0}^N c_k^2 + \sum_{k=0}^N d_k^2} \frac{1}{(4\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{ds dt ds' dt'}{\sqrt{(\sum_{k=0}^N c_k^2 + \sum_{k=0}^N d_k^2)^2 - (F(s, t, s', t'))^2}},$$

where $F(s, t, s', t') = \mathbf{r}_2(s) \cdot \mathbf{r}_2(s') + \mathbf{r}_1(t) \cdot \mathbf{r}_1(t')$. If we consider again the limit as $n \rightarrow \infty$, then we will have:

$$\langle \text{Lk}^2 \rangle \leq \frac{4}{\pi} \frac{\|\mathbf{c}'\|_{l^2}^2 \|\mathbf{d}'\|_{l^2}^2}{\|\mathbf{c}\|_{l^2}^2 + \|\mathbf{d}\|_{l^2}^2} \frac{1}{(4\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{ds dt ds' dt'}{\sqrt{(\|\mathbf{c}\|_{l^2}^2 + \|\mathbf{d}\|_{l^2}^2)^2 - (F(s, t, s', t'))^2}},$$

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Further simplifying, if we take $c_k = d_k = \frac{1}{k^\alpha}$, then $F(s, t, s', t')$ can be written as a sum of polylogarithms. Moreover, it's clear that $\langle \text{Lk}^2 \rangle$ will therefore be bounded when C is and when the sequence \mathbf{c} decays faster than $n^{(-3-\epsilon)/2}$.

Chapter 4

Second Moments for Higher Dimensional Linking Integrals

4.1 Integration Over Codimension-1 Subspaces

As in [18], given two closed, disjoint, oriented manifolds M_1 and M_2 of respective dimensions m and n which are submanifolds of $\mathbb{R}^{N=m+n+1}$, then one may generalize the classic Gauss linking integral as:

$$\text{Lk}(M_1, M_2) = \frac{(-1)^{m+1}}{\text{vol}(\mathbb{S}^{N-1})} \int_{M_1 \times M_2} \frac{\text{Det}(\mathbf{x} - \mathbf{y}, \frac{d\mathbf{x}}{ds_1}, \dots, \frac{d\mathbf{x}}{ds_m} \frac{d\mathbf{y}}{dt_1}, \dots, \frac{d\mathbf{y}}{dt_n})}{\|\mathbf{x} - \mathbf{y}\|^N} ds_1 \dots ds_m dt_1 \dots dt_n, \quad (4.1.1)$$

where $\mathbf{x}(s_1, \dots, s_m)$ and $\mathbf{y}(t_1, \dots, t_n)$ are local coordinates on the manifolds M_1 and M_2 . This way, if we take as starting data two manifolds in $\mathbb{R}^{N'}$, where $N' = m + n + 2$, then we may generate a random link of manifolds in \mathbb{R}^N by picking an $N = m + n + 1$ dimensional subspace of $\mathbb{R}^{N'}$ at random, and then orthogonally projecting the manifolds to the subspace. To compute the second moment of the linking number, we may mimic the method in the previous section almost line for line, except for the computation of the I_{ii} integrals, and for these we will need to make an assumption, namely that $m + n + 2$ is even. Now let M_1 and M_2 be manifolds in $\mathbb{R}^{N'}$ with local coordinates $\mathbf{x}(s_1, s_2, \dots, s_m)$ and $\mathbf{y}(t_1, t_2, \dots, t_n)$. As above, to make the notation more compact, we will make the following conventions:

$$\begin{aligned} \mathbf{a}_i(s_1, s_2, \dots, s_m) &= \frac{\partial \mathbf{x}(s_1, s_2, \dots, s_m)}{\partial s_i} \text{ for } 1 \leq i \leq m \\ \mathbf{a}_j(t_1, t_2, \dots, t_n) &= \frac{\partial \mathbf{y}(t_1, t_2, \dots, t_n)}{\partial t_{j-m}} \text{ for } m+1 \leq j \leq m+n \\ \mathbf{a}_{m+n+1}(s_1, s_2, \dots, s_m, t_1, t_2, \dots, t_n) &= \mathbf{y}(t_1, \dots, t_n) - \mathbf{x}(s_1, \dots, s_m) \end{aligned}$$

with similar identifications for the vectors \mathbf{a}'_i . Also define that:

$$[\mathbf{a}_1 : \mathbf{a}_2 : \dots : \mathbf{a}_N]_i = \sum_{j_1, j_2, \dots, j_N=1}^{N+1} \epsilon_{ij_1 j_2 \dots j_N} a_{1j_1} a_{2j_2} \dots a_{Nj_N}$$

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We will now prove the following:

Theorem 4.1.1. *Given two closed, disjoint, oriented manifolds M_1 and M_2 of respective dimensions m and n in \mathbb{R}^{m+n+2} , where $m+n$ is even, then the value of Lk^2 averaged over all orthogonal projections to $m+n+1$ dimensional subspaces, is given by the following integral:*

$$\frac{1}{\text{vol}(\mathbb{S}^{N-1})^2} \int_{M_1 \times M_2} I_{\langle \text{Lk}^2 \rangle}(s_1, \dots, s_m, t_1, \dots, t_n) ds_1 \dots ds_m dt_1 \dots dt_n$$

where

$$\begin{aligned} I_{\langle \text{Lk}^2 \rangle}(A, A') & \| \mathbf{a}_N \|^N \| \mathbf{a}'_N \|^N = \\ & I_{1,1}(a, b) ([\mathbf{b}_2 : \mathbf{b}_3 : \dots : \mathbf{b}_{m+n+2}] \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \dots : \mathbf{a}_{m+n+1}]) ([\mathbf{b}_2 : \mathbf{b}_3 : \dots : \mathbf{b}_{m+n+2}] \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \dots : \mathbf{a}'_{m+n+1}]) + \\ & I_{2,2}(a, b) ([\mathbf{b}_1 : \mathbf{b}_3 : \dots : \mathbf{b}_{m+n+2}] \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \dots : \mathbf{a}_{m+n+1}]) ([\mathbf{b}_1 : \mathbf{b}_3 : \dots : \mathbf{b}_{m+n+2}] \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \dots : \mathbf{a}'_{m+n+1}]) + \\ & I_{3,3}(a, b) \sum_{i=3}^{m+n+2} \mathbf{b}_i \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \dots : \mathbf{a}_{m+n+1}] \mathbf{b}_i \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \dots : \mathbf{a}'_{m+n+1}] \end{aligned}$$

and $\{\mathbf{b}_i\}_{i=1, \dots, m+n+2}$ is a basis for \mathbb{R}^{m+n+2} such that $\mathbf{b}_1, \mathbf{b}_2 \in \text{span}\{\mathbf{a}_{m+n+1}, \mathbf{a}'_{m+n+1}\}$ and $I_{3,3}(m+n)$ is a function of $a = \mathbf{b}_1 \cdot \frac{\mathbf{a}_{m+n+1}}{\|\mathbf{a}_{m+n+1}\|}$ and $b = \mathbf{b}_2 \cdot \frac{\mathbf{a}'_{m+n+1}}{\|\mathbf{a}'_{m+n+1}\|}$.

PROOF Take $A = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_N)$, $A' = (\mathbf{a}'_1, \mathbf{a}'_2, \dots, \mathbf{a}'_N)$, $dV = \exp(-\|\mathbf{v}\|^2/2) d\mathbf{v}$, and define:

$$I_{\langle \text{Lk}^2 \rangle}(A, A') = \frac{1}{(2\pi)^{N'/2}} \int_{\mathbb{R}^{N'}} \frac{\text{Det}((\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_N, v/\|v\|))}{\|\text{proj}_{v^\perp}(\mathbf{a}_N)\|^N} \frac{\text{Det}((\mathbf{a}'_1, \mathbf{a}'_2, \dots, \mathbf{a}'_N, v/\|v\|))}{\|\text{proj}_{v^\perp}(\mathbf{a}'_N)\|^N} dV.$$

As in the previous section, define that:

$$f(A, A') = \frac{\det(A^T A)^{1/2} \det(A'^T A')^{1/2}}{\|\mathbf{a}_N\|^N \|\mathbf{a}'_N\|^N}$$

so that by applying Gram-Schmidt we obtain:

$$I_{\langle \text{Lk}^2 \rangle}(A, A') = \frac{f(A, A')}{(2\pi)^{N'/2}} \int_{\mathbb{R}^{N'}} \frac{\text{Det}(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N, v/\|v\|)}{\|\text{proj}_{v^\perp}(\mathbf{q}_N)\|^N} \frac{\text{Det}((\mathbf{q}'_1, \mathbf{q}'_2, \dots, \mathbf{q}'_N, v/\|v\|))}{\|\text{proj}_{v^\perp}(\mathbf{q}'_N)\|^N} dV$$

Similar to the argument in the second section, choose a basis such that:

$$\begin{aligned} \mathbf{b}_1 &= \frac{\mathbf{q}_{m+n+1} + \mathbf{q}'_{m+n+1}}{\|\mathbf{q}_{m+n+1} + \mathbf{q}'_{m+n+1}\|} \\ \mathbf{b}_2 &= \frac{\mathbf{q}_{m+n+1} - \mathbf{q}'_{m+n+1}}{\|\mathbf{q}_{m+n+1} - \mathbf{q}'_{m+n+1}\|} \\ \text{and } \mathbf{b}_3, \dots, \mathbf{b}_{m+n+2} &\in \text{span}^\perp\{\mathbf{b}_1, \mathbf{b}_2\} \end{aligned}$$

This choice is made to simplify the denominators in $I(A, A')$. The relevant Gaussian integrals that appear in computing $I(A, A')$ upon expanding the product of the determinants and

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integrating are of the form:

$$I_{i,j} = \frac{1}{(2\pi)^{N'/2}} \int_{\mathbb{R}^{N'}} \frac{\|\mathbf{v}\|^{2(m+n)} v_i v_j dV}{((bv_1 - av_2)^2 + v_3^2 + \dots + v_{N'}^2)^{N/2} ((bv_1 + av_2)^2 + v_3^2 + \dots + v_{N'}^2)^{N/2}}$$

Expanding the determinants we have:

$$\begin{aligned} \frac{I_{\langle \text{Lk}^2 \rangle}(A, A')}{f(A, A')} = & I_{1,1}(a, b)([\mathbf{b}_2 : \mathbf{b}_3 : \dots : \mathbf{b}_{m+n+2}] \cdot [\mathbf{q}_1 : \mathbf{q}_2 : \dots : \mathbf{q}_{m+n+1}])([\mathbf{b}_2 : \mathbf{b}_3 : \dots : \mathbf{b}_{m+n+2}] \cdot [\mathbf{q}'_1 : \mathbf{q}'_2 : \dots : \mathbf{q}'_{m+n+1}]) + \\ & I_{2,2}(a, b)([\mathbf{b}_1 : \mathbf{b}_3 : \dots : \mathbf{b}_{m+n+2}] \cdot [\mathbf{q}_1 : \mathbf{q}_2 : \dots : \mathbf{q}_{m+n+1}])([\mathbf{b}_1 : \mathbf{b}_3 : \dots : \mathbf{b}_{m+n+2}] \cdot [\mathbf{q}'_1 : \mathbf{q}'_2 : \dots : \mathbf{q}'_{m+n+1}]) + \\ & \sum_{i=3}^{m+n+2} I_{i,i}(a, b) \mathbf{b}_i \cdot [\mathbf{q}_1 : \mathbf{q}_2 : \dots : \mathbf{q}_{m+n+1}] \mathbf{b}_i \cdot [\mathbf{q}'_1 : \mathbf{q}'_2 : \dots : \mathbf{q}'_{m+n+1}] \end{aligned}$$

In a similar fashion to the case considered in the previous chapter, the terms including $I_{1,1}$ and $I_{2,2}$ cancel out precisely when $m+n=2$. When $m+n>2$ no clear cancellation occurs. Moreover, it is clear that $I_{i,j}=0$ for $i \neq j$ and $I_{3,3}=I_{4,4}=\dots=I_{m+n+2,m+n+2}$. With these facts we may simplify the above equation to write:

$$\begin{aligned} I_{\langle \text{Lk}^2 \rangle}(A, A') \|\mathbf{a}_N\|^N \|\mathbf{a}'_N\|^N = & I_{1,1}(a, b)([\mathbf{b}_2 : \mathbf{b}_3 : \dots : \mathbf{b}_{m+n+2}] \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \dots : \mathbf{a}_{m+n+1}])([\mathbf{b}_2 : \mathbf{b}_3 : \dots : \mathbf{b}_{m+n+2}] \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \dots : \mathbf{a}'_{m+n+1}]) + \\ & I_{2,2}(a, b)([\mathbf{b}_1 : \mathbf{b}_3 : \dots : \mathbf{b}_{m+n+2}] \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \dots : \mathbf{a}_{m+n+1}])([\mathbf{b}_1 : \mathbf{b}_3 : \dots : \mathbf{b}_{m+n+2}] \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \dots : \mathbf{a}'_{m+n+1}]) + \\ & I_{3,3}(a, b) \sum_{i=3}^{m+n+2} \mathbf{b}_i \cdot [\mathbf{a}_1 : \mathbf{a}_2 : \dots : \mathbf{a}_{m+n+1}] \mathbf{b}_i \cdot [\mathbf{a}'_1 : \mathbf{a}'_2 : \dots : \mathbf{a}'_{m+n+1}] \end{aligned}$$

□

Now we will focus on the functional form of the function $I_{3,3}(a, b)$, and to do so, we will reduce it to an integral over a 3-sphere as in the previous section. We have that:

$$I_{3,3} = \frac{1}{(2\pi)^{N'/2}} \int_{\mathbb{R}^{N'}} \frac{\|\mathbf{v}\|^{2(m+n)} v_3^2 dV}{((bv_1 - av_2)^2 + v_3^2 + \dots + v_{N'}^2)^{N/2} ((bv_1 + av_2)^2 + v_3^2 + \dots + v_{N'}^2)^{N/2}}$$

Now change the coordinates $v_4, v_5, \dots, v_{m+n+2}$ to $(m+n-1)$ -dimensional spherical coordinates so that $v_4^2 + v_5^2 + \dots + v_{m+n+2}^2 = r^2$ and $dv_4 dv_5 \dots dv_{m+n+2} = r^{m+n-2} dr dv_{\text{vol}}(\mathbb{S}^{m+n-2})$ so that the above integral becomes:

$$I_{3,3} = A_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{(v_1^2 + v_2^2 + v_3^2 + r^2)^{(m+n)} v_3^2 dV(r)}{((bv_1 - av_2)^2 + v_3^2 + r^2)^{N/2} ((bv_1 + av_2)^2 + v_3^2 + r^2)^{N/2}}$$

where $dV(r) = \exp(-(v_1^2 + v_2^2 + v_3^2 + r^2)/2) r^{m+n-2} dr dv_1 dv_2 dv_3$ and $A_1 = \frac{\text{vol}(\mathbb{S}^{N'-4})}{(2\pi)^{N'/2}}$. Now make the change of coordinates $r \rightarrow v_4$, and realize that since $m+n+2$ was chosen to be

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even, then the whole integrand is even and we obtain that:

$$I_{3,3} = \frac{A_1}{2ab} \int_{\mathbb{R}^4} \frac{(v_1^2/b^2 + v_2^2/a^2 + v_3^2 + v_4^2)^{(m+n)} v_3^2 v_4^{m+n-2}}{((v_1 - v_2)^2 + v_3^2 + v_4^2)^{N/2} ((v_1 + v_2)^2 + v_3^2 + v_4^2)^{N/2}} dV'$$

where again we have that $a = \mathbf{q}_N \cdot \mathbf{b}_1$ and $b = \mathbf{q}_N \cdot \mathbf{b}_2$ and used a further change of coordinates: $v_1 \rightarrow v_1/b$, and $v_2 \rightarrow v_2/a$ and have written that $dV' = \exp(-(v_1^2/a^2 + v_2^2/b^2 + v_3^2 + v_4^2)/2)$. Finally, change to toroidal coordinates to obtain:

$$I_{3,3} = \frac{A_1}{2ab} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \frac{k_1^{m+n} \sin^N(\sigma) \cos(\sigma) \sin^{N'-4}(\phi) \cos^2(\phi) dV'}{(1 - \cos^4(\sigma) \sin^2(2\theta))^{N/2}}$$

where again $k_1(\theta, \sigma) = \sin^2(\sigma) + \cos^2(\sigma)((\cos(\theta)/b)^2 + (\sin(\theta)/a)^2)$ and moreover $dV' = \exp(-k_1 r^2/2) r^{m+n+1} dr d\theta d\phi d\sigma$. To integrate out the radial dependence, first change $r \rightarrow \frac{r}{\sqrt{k_1}}$:

$$I_{3,3} = \frac{A_1}{2ab} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \frac{k_1^{(N'-4)/2} \sin^N(\sigma) \cos(\sigma) \sin^{N'-4}(\phi) \cos^2(\phi) \exp(-r^2/2) dV'}{(1 - \cos^4(\sigma) \sin^2(2\theta))^{N/2}}$$

where now $dV' = r^N dr d\theta d\phi d\sigma$. Notice that the for the case of links when $m = n = 1$ that this integral is especially easy. Now integrate over the radial coordinate:

$$I_{3,3} = A_2 \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2\pi} \frac{k_1^{(N'-4)/2} \sin^N(\sigma) \cos(\sigma) \sin^{N'-4}(\phi) \cos^2(\phi)}{(1 - \cos^4(\sigma) \sin^2(2\theta))^{N/2}} d\theta d\phi d\sigma$$

where the new constant A_2 is given by:

$$A_2 = \frac{\text{vol}(\mathbb{S}^{m+n-2}) \left(\frac{m+n}{2}\right)! 2^{(m+n)/2}}{2ab(2\pi)^{(m+n+2)/2}}$$

Lastly, define:

$$I(\theta, \phi, \sigma, m, n) = A_2 \frac{k_1^{(N'-4)/2} \sin^N(\sigma) \cos(\sigma) \sin^{N'-4}(\phi) \cos^2(\phi)}{(1 - \cos^4(\sigma) \sin^2(2\theta))^{N/2}}.$$

Given the sufficiently simple form above, we may then compute some values of $I_{3,3}(m+n)$ for different values of m and n such that $m+n$ is even. Here is a list of a few of them, calculated using a symbolical integration in Wolfram Mathematica:

$$\begin{aligned} I_{3,3}(m+n=2) &= \frac{1}{\pi ab} \text{ (see section 2)} \\ I_{3,3}(m+n=4) &= \frac{1 + 4a^2b^2}{9\pi a^3b^3} \\ I_{3,3}(m+n=6) &= \frac{9b^4 + 2a^2b^2(5 + 16b^2) + a^4(9 + 32b^2 + 128b^4)}{450\pi a^5b^5} \end{aligned}$$

4.1. Integration Over Codimension-1 Subspaces

$$\begin{aligned}
& I_{3,3}(m+n=8) \\
&= \frac{15b^6 + 3a^2b^4(7 + 20b^2) + 3a^6(1 + 4b^2)(5 + 64b^4) + a^4b^2(21 + 56b^2 + 192b^4)}{3675\pi a^7 b^7}
\end{aligned}$$

Notice that this computation is very similar to computation in 3.1.2, and uses the fact that the subspaces being projected on to are codimension 1 so that the Grassmannian is identified with a sphere. With this identification we were able to compute $\langle \text{Lk}^2(M, N) \rangle$ using the Lebesgue measure on the sphere, however, this assumption on the codimension can be removed with an argument very much similar to that in 3.2.1. Given this computation, it is then feasible to bound $\langle \text{Lk}(M, N)^2 \rangle$ by bounding the configuration space integrals in a way analogous to the method at the end of section 3.3.

Remark 4.1.2. It would be interesting to calculate the values of the integral $I_{i,i}(a, b)$ when $m+n$ is odd as well. For example, in the case when $m+n=1$, the results could be used to find the second moment of the winding number for random plane curves sampled in a way analogous to the method in the previous chapters (that is, by projecting space curves in \mathbb{R}^n on to subspaces sampled from $Gr(n, 2)$ instead of $Gr(n, 3)$).

Chapter 5

Numerical Study: Petal Diagrams

5.1 Model Details

In this chapter, we will explore a very special case of the input curve $\mathbf{r}(t)$, inspired by [11] in order to show how the results in the previous chapters may be used. The model considered in their paper is called the Petaluma model and is motivated by the observation in [1] that an embedding of a knot may be arranged so that there is a projection to a plane $P = \mathbf{v}^\perp$ where the knot diagram obtained is a rose with n petals. One may enhance this diagram to account for the crossing data by labeling the strands with the heights through which they pass through the axis determined by \mathbf{v} . With this observation in hand, the Petaluma model is defined by fixing a petal diagram and then choosing permutations of the heights at random.

We will now focus on finding a space curve $\mathbf{r}(t)$ in some \mathbb{R}^N such that our random projection model can approximate the Petaluma model. To do so, assume k is odd (the even case follows similarly) and subdivide the interval $[0, \pi]$ into $2k$ equal length subintervals, $\{U_i = [t_i, t_{i+1}]\}$, and define a space curve $\mathbf{r}(t) : \mathbb{S}^1 \rightarrow \mathbb{R}^{2+k}$ such that:

$$\mathbf{r}(t, \epsilon, k) = \cos(kt) \cos(t) \mathbf{e}_1 + \cos(kt) \sin(t) \mathbf{e}_2 + \mathbf{R}(t, \epsilon, k) \quad (5.1.1)$$

where:

$$\mathbf{R}(t, \epsilon, k) = \sum_{i=1}^k \left(\mathbb{1}_{U_{2i-1}}(t) \frac{2k\epsilon(t - t_i)}{\pi} + \mathbb{1}_{U_{2i}}(t) \frac{2k\epsilon(t_{i+1} - t)}{\pi} \right) \mathbf{e}_{i+2} \quad (5.1.2)$$

and $\mathbb{1}_{U_i}$ is the indicator function on the interval U_i . Intuitively, in the first two coordinates we have the parametrization for a rose with k petals, and in the remaining k coordinates we have a linear function that for the i^{th} strand is supported in the $(2+i)th$ coordinate and runs from 0 at the outermost part of the strand to ϵ at the center and then back to 0. For the case of polygonal knots we may take a piecewise linear approximation of the rose diagram in the first two coordinates. The piecewise linear petal diagram is related to the grid model also considered in [11].

5.2. Expected Total Curvature

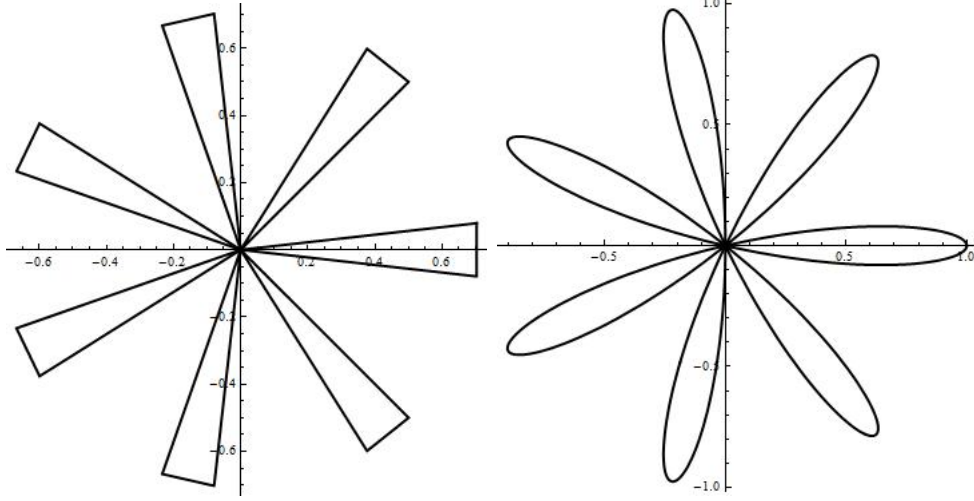


Figure 5.1.1: Petal diagrams in the first 2 coordinates. The left diagram is used in the PL case, while the right is used in our approximation of the Petaluma model.

5.2 Expected Total Curvature

We will be interested in computing the expected curvature in both cases, the latter of which is calculated as the sum of the turning angles. Given this particularly simple form for $\mathbf{r}(t)$, we then have that:

$$\|\mathbf{r}'(t, \epsilon, k)\|^2 = \frac{4\epsilon^2 k^2}{\pi^2} + \frac{1}{2}((1 + k^2) - (k^2 - 1)\cos(2kt))$$

$$\|\mathbf{r}''(t, \epsilon, k)\|^2 = \frac{1}{2}(1 + 6k^2 + k^4 + (k^2 - 1)^2 \cos(2kt))$$

$$(\mathbf{r}'(t, \epsilon, k) \cdot \mathbf{r}''(t, \epsilon, k))^2 = \frac{1}{4}k^2(k^2 - 1)^2 \sin(2kt)^2$$

$$I_{\langle \kappa(C) \rangle}(\mathbf{r}'(t, \epsilon, k), \mathbf{r}''(t, \epsilon, k)) = \frac{\sqrt{\|\mathbf{r}'(t, \epsilon, k)\|^2 \|\mathbf{r}''(t, \epsilon, k)\|^2 - (\mathbf{r}'(t, \epsilon, k) \cdot \mathbf{r}''(t, \epsilon, k))^2}}{\|\mathbf{r}'(t, \epsilon, k)\|^2}$$

For small ϵ , this integral will get closer and closer to simply computing the curvature of the petal diagram. Numerical calculations show that $\int_0^\pi I_{\langle \kappa(C) \rangle}(\mathbf{r}'(t, 0, k), \mathbf{r}''(t, 0, k)) = \langle \kappa(C) \rangle \approx \pi(k + 1)$, and for small enough values of epsilon we have:

$$\int_0^\pi I_{\langle \kappa(C) \rangle}(\mathbf{r}'(t, \epsilon, k), \mathbf{r}''(t, \epsilon, k)) \leq \pi(k + 1) \quad (5.2.1)$$

Interestingly, in the case when $k = 3$ and $\epsilon = .5$ (though this can be further tuned), then we may get an idea about the approximate density of the unknot. That is, exactly like in the proof of corollary 25 in [6] (also see [5]) which uses the Fary-Milnor theorem ([16]), if we let x denote the fraction of knots with curvature greater than 4π , then

$$\langle \kappa(C) \rangle > 4\pi x + 2\pi(1 - x)$$

5.3. $\langle \text{Lk}^2 \rangle$

and solving for x we see that:

$$x < \frac{\langle \kappa(C) \rangle}{2\pi} - 1$$

When $\epsilon = .5$ a numerical integration gives that $\langle \kappa(C) \rangle \approx 9.72$ so that $x < .54$, that is, at least approximately 48 percent are unknotted.

Remark 5.2.1. The same results holds when considering a piecewise linear approximation to the petal diagram as discussed in the above remark. It would be interesting to compute the total torsion as well, especially given its relation to the self-linking number.

5.3 $\langle \text{Lk}^2 \rangle$

We may modify the curve $\mathbf{r}(t)$ defined in the first section to define a model for random links. To so, we define the following two space curves:

$$\begin{aligned} \mathbf{r}_1(t, \epsilon, k) &= \cos(kt) \cos(t) \mathbf{e}_1 + \cos(kt) \sin(t) \mathbf{e}_2 + \mathbf{R}(t, \epsilon, k) \\ \mathbf{r}_2(t', \epsilon, k) &= \begin{pmatrix} \cos(\frac{k-2}{k}\pi) & -\sin(\frac{k-2}{k}\pi) \\ \sin(\frac{k-2}{k}\pi) & \cos(\frac{k-2}{k}\pi) \end{pmatrix} \begin{pmatrix} \cos(kt') \cos(t') \\ \cos(kt') \sin(t') \end{pmatrix} + \mathbf{R}(t', \epsilon, k) \end{aligned}$$

in \mathbb{R}^{2+2k} , again where $\mathbf{R}(t', \epsilon, k)$ is as in (5.1.2), and k zeroes are appended to the end of the vector $\mathbf{r}_1(t)$, and where another k zeros are inserted after the second position in $\mathbf{r}_2(t)$. That is, we take each component to have an equal number of petals and the second component is a rotation of the first in the first two coordinates, as is shown in figure 5.3.1.

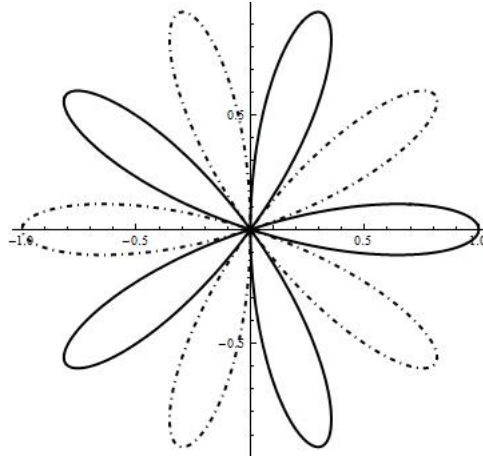


Figure 5.3.1: Petal diagram with $k = 5$ petals per component.

In tables 5.1 and 5.2 we illustrated two ways of computing $\langle \text{Lk}^2 \rangle$, the first of which was found by sampling 10,000 links using our model (subspaces were chosen by applying the Gram-Schmidt process to n by 3 matrices whose entries are uniformly distributed in $(-1, 1)$), computing the linking number for each sample by taking a piecewise linear approximation of the link and then using the algorithm in [3] to compute the linking number, and then finding the sum of the squared standard deviation and the squared mean of our samples. The second

5.3. $\langle \text{Lk}^2 \rangle$

was found by computing the integral in Theorem 3.2.1 using a Monte Carlo integration in Wolfram Mathematica.

$\langle \text{Lk}^2 \rangle$ Comparison with $\epsilon = 1$		
k (number of petals/ component)	Sampled $\langle \text{Lk}^2 \rangle$	Monte Carlo $\langle \text{Lk}^2 \rangle$
3	.6042	.6664
5	1.6739	1.7171
7	3.1945	3.1475
9	5.1417	5.2137

Table 5.1: $\langle \text{Lk}^2 \rangle$ Comparison with $\epsilon = 1$

$\langle \text{Lk}^2 \rangle$ Comparison with $\epsilon = .1$		
k (number of petals/ component)	Sampled $\langle \text{Lk}^2 \rangle$	Monte Carlo $\langle \text{Lk}^2 \rangle$
3	.5269	.4834
5	1.7108	1.3848
7	3.5662	2.7253
9	6.0351	-

Table 5.2: $\langle \text{Lk}^2 \rangle$ Comparison with $\epsilon = .1$

Remark 5.3.1. Notice that in the second table that the values of $\langle \text{Lk}^2 \rangle$ computed with the two methods appear to be quite different. We expect this is due to the numerical instability that arises when ϵ is small since the strands become closer and force the denominator in the expression for $I_{\langle \text{Lk}^2 \rangle}$ to become close to zero, unlike in the results in the first table. It would be interesting to find a stable numerical method for computing these configuration space integrals. Alternate methods are explored in the appendix.

Chapter 6

Future Directions

In this work we have found a way to compute the second moment of the linking number for a rather general model of random links. Of course, it would be interesting to find the moments of other knot and link invariants arising from configuration space integrals. One such example, the writhe of a knot, although not a knot invariant, provides some insight in to the steps that would be involved. To compute the writhe, one starts with a differentiable closed curve $\gamma(t)$ in \mathbb{R}^3 , and then computes an analogue of the Gauss linking integral:

$$\text{Wr}(\gamma) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{(\dot{\mathbf{r}}_1(t) \times \dot{\mathbf{r}}_2(t')) \cdot (\mathbf{r}_2(t') - \mathbf{r}_1(t))}{\|\mathbf{r}_2(t') - \mathbf{r}_1(t)\|^3} dt' dt \quad (6.0.1)$$

The form of this integral is especially well suited to the analysis in chapter 3, and in fact an almost identical result would be obtained, however some care would need to be taken in proving that the order of integration may be reversed due to the singularity in the integrand in (6.0.1).

After understanding the singularities involved in computing $\langle \text{Wr}^2 \rangle$, then other knot invariants that may be computed with configuration space integrals (see [20]) may be explored. For example, in [15] they discuss a knot invariant, $v_2 = I_X(\gamma) - I_Y(\gamma)$ for a smooth embedding $\gamma(t) : \mathbb{S}^1 \rightarrow \mathbb{R}^3$, by integrating over the configuration spaces:

$$\Delta_4 = \{(t_1, t_2, t_3, t_4) | 0 < t_1 < t_2 < t_3 < t_4 < 1\}, \text{ and} \\ \Delta_3 = \{(t_1, t_2, t_3, z) | 0 < t_1 < t_2 < t_3 < 1, \mathbf{z} \in \mathbb{R}^3 - \{(\gamma(t_1), \gamma(t_2), \gamma(t_3))\}\},$$

where:

$$I_X(\gamma) = -\frac{1}{(4\pi)^2} \int_{\Delta_4} \frac{\text{Det}[\gamma(t_3) - \gamma(t_1), \gamma'(t_3), \gamma'(t_1)]}{\|\gamma(t_3) - \gamma(t_1), \gamma'(t_3)\|^3} \frac{\text{Det}[\gamma(t_4) - \gamma(t_2), \gamma'(t_4), \gamma'(t_2)]}{\|\gamma(t_4) - \gamma(t_2)\|^3} dt_1 dt_2 dt_3 dt_4$$

and

$$I_Y(\gamma) = -\frac{1}{(4\pi)^3} \int_{\Delta_3(\gamma)} \text{Det}[E(\mathbf{z}, t_1), E(\mathbf{z}, t_2), E(\mathbf{z}, t_3)] d\mathbf{z} dt_1 dt_2 dt_3 \\ \text{with } E(\mathbf{z}, t) = \frac{(\mathbf{z} - \gamma(t)) \times \gamma'(t)}{\|\mathbf{z} - \gamma(t)\|^3}.$$

It would be interesting to compute quantities like $\langle v_2 \rangle$ and $\langle v_2^2 \rangle$ for the model of random knots we have considered in this work. Notice that $I_X(\gamma)$ has similar tensorial properties as Lk^2 and so we may compute $\langle I_X(\gamma) \rangle$ by setting

$$\begin{aligned} \mathbf{a}_1(t_1) &= \gamma'(t_1) , \mathbf{a}_2(t_3) = \gamma'(t_3) , \mathbf{a}_3(t_3, t_1) = \gamma(t_3) - \gamma(t_1) \\ \mathbf{a}'_1(t_2) &= \gamma'(t_2) , \mathbf{a}'_2(t_4) = \gamma'(t_4) , \mathbf{a}'_3(t_2, t_4) = \gamma(t_4) - \gamma(t_2) \\ A(t_1, t_3) &= [\mathbf{a}_1(t_1), \mathbf{a}_2(t_3), \mathbf{a}_3(t_1, t_3)] \text{ and } A'(t_2, t_4) = [\mathbf{a}'_1(t_2), \mathbf{a}'_2(t_4), \mathbf{a}'_3(t_2, t_4)] \end{aligned}$$

so that:

$$\langle I_X(\gamma) \rangle = -\frac{1}{(4\pi)^2} \int_{\Delta_4} I_{\langle \text{Lk}^2 \rangle}(A(t_1, t_3), A'(t_2, t_4)) dt_1 dt_2 dt_3 dt_4 \quad (6.0.2)$$

where $I_{\langle \text{Lk}^2 \rangle}(A, A')$ is the same function as in the main result (3.1.6). We could also find $\langle I_X(\gamma)^2 \rangle$ if we knew $\langle \text{Lk}^4 \rangle$. It would also be interesting to find $\langle I_Y(\gamma) \rangle$ and $\langle I_Y(\gamma)^2 \rangle$, however the Gaussian integrals involved here are considerably more difficult.

Remark 6.0.2. Notice that if instead of considering random projections of a fixed space curve to 3 dimensional subspaces we considered projections to 2 dimensional subspaces, then we would obtain a model of random plane curves. In [15] it is shown that for immersions $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ that $I_X(\gamma) = 0$ and that $I_Y(\gamma)$ is an invariant of plane curves, which motivates finding quantities like $\langle I_Y(\gamma) \rangle$ and $\langle I_Y(\gamma)^2 \rangle$. Notice that in the model considered in the previous section, that as ϵ approaches 0, then the distribution of the permutations of the heights seems to approach the uniform distribution, and also the projections become closer to being planar. In this way, the distribution of $I_Y(\gamma)$ should approach the distribution of v_2 .

Beyond invariants of knots and links in \mathbb{R}^3 , there are also configuration space integrals for computing linking numbers of links in hyperbolic space, for which a computation akin to the in chapter 3 seems both interesting and feasible, provided one found a reasonable integral-geometric object like $Gr(n, 3)$ to integrate over.

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Appendix A

Multidegree Method

In chapter 3 we found a closed form for the function $I_{\langle \text{Lk}^2 \rangle}(A, A')$. In this section we will study how one may find the function $I_{\langle \text{Lk}^2 \rangle}(A, A')$ numerically by making an assumption on its functional form. To this end, define that $\mathbf{A}_i = \mathbf{a}_i$ for $1 \leq i \leq 3$ and $\mathbf{A}_i = \mathbf{a}'_{i \bmod 3}$ for $4 \leq i \leq 6$ then by 3.1.6 we have actually that:

$$I(A, A') = f(\mathbf{x})/g(\mathbf{x}) \quad (\text{A.0.1})$$

where $f(\mathbf{x}), g(\mathbf{x})^2 \in \mathbb{R}[\{\mathbf{A}_i \cdot \mathbf{A}_j\}_{i,j \in \{1,2,3,4,5,6\}}]$. That is, if we consider the polynomial ring $\mathbb{R}[\{\mathbf{A}_i \cdot \mathbf{A}_j\}_{i,j \in \{1,2,3,4,5,6\}}]$ generated by the pairwise inner products of the vectors defined by the initial space curves, then the function $I_{\langle \text{Lk}^2 \rangle}(A, A')$ is a ratio of a polynomial $f(\mathbf{x})$ and the square root of a polynomial $g(\mathbf{x})$. Moreover, we may put a grading on this polynomial ring, $\nu : \mathbb{R}[\{\mathbf{A}_i \cdot \mathbf{A}_j\}_{i,j \in \{1,2,3,4,5,6\}}] \rightarrow (\mathbb{Z}^+)^6$ that simply counts the number of occurrences of the vectors \mathbf{a}_i . For example,

$$\nu((\mathbf{a}_1 \cdot \mathbf{a}_2)(\mathbf{a}'_2 \cdot \mathbf{a}'_1)(\mathbf{a}_3 \cdot \mathbf{a}'_3)(\mathbf{a}_3 \cdot \mathbf{a}_3)(\mathbf{a}_3 \cdot \mathbf{a}_3)) = (1, 1, 3, 1, 1, 3) \quad (\text{A.0.2})$$

In this way, we see that $f(\mathbf{x})$ is in the $(1, 1, 3, 1, 1, 3)$ graded part, while $g(\mathbf{x})^2$ is in the $(0, 0, 10, 0, 0, 10)$ part. Notice that given the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3\}$ in \mathbb{R}^n , they generically span a 6 dimensional subspace, and so we will be interested in finding $I_{\langle \text{Lk}^2 \rangle}(A, A')$ when the vectors are in \mathbb{R}^6 . Much like in the previous sections, we set $U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ and $dU = e^{-\frac{1}{2}(\|\mathbf{u}_1\|^2 + \|\mathbf{u}_2\|^2 + \|\mathbf{u}_3\|^2)} d\mathbf{u}_1 d\mathbf{u}_2 d\mathbf{u}_3$. Now assume we knew that the function:

$$I(A, A') = \frac{1}{(2\pi)^9} \int_{(\mathbb{R}^6)^3} \frac{\text{Det}([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]) \text{Det}([\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3])}{\text{Det}(U^T U) \|\text{proj}_{U^\perp}(\mathbf{a}_3)\|^3 \|\text{proj}_{U^\perp}(\mathbf{a}'_3)\|^3} dU$$

was of the form $\frac{f(\mathbf{x})}{g(\mathbf{x})}$ where $f(\mathbf{x}), g(\mathbf{x})^2 \in \mathbb{R}[\{\mathbf{A}_i \cdot \mathbf{A}_j\}_{i,j \in \{1,2,3,4,5,6\}}]$ and $\nu(f(\mathbf{x})) = (1, 1, 3, 1, 1, 3)$ and $\nu(g(\mathbf{x})^2) = (0, 0, 10, 0, 0, 10)$, then we could attempt to find the function $I(A, A')$ by using the various symmetries of Lk to find $f(\mathbf{x})$ and $g(\mathbf{x})^2$. To do so, first notice that the $(0, 0, 10, 0, 0, 10)$ graded part is generated by the monomials:

$$P_1 = \|\mathbf{a}_3\|^{10} \|\mathbf{a}'_3\|^{10}, P_2 = \|\mathbf{a}_3\|^8 \|\mathbf{a}'_3\|^8 (\mathbf{a}_3 \cdot \mathbf{a}'_3)^2, P_3 = \|\mathbf{a}_3\|^6 \|\mathbf{a}'_3\|^6 (\mathbf{a}_3 \cdot \mathbf{a}'_3)^4 \\ P_4 = \|\mathbf{a}_3\|^4 \|\mathbf{a}'_3\|^4 (\mathbf{a}_3 \cdot \mathbf{a}'_3)^6, P_5 = \|\mathbf{a}_3\|^2 \|\mathbf{a}'_3\|^2 (\mathbf{a}_3 \cdot \mathbf{a}'_3)^8, P_6 = (\mathbf{a}_3 \cdot \mathbf{a}'_3)^{10}$$

With respect to this basis we have $g(\mathbf{x})^2 = \sum_{i=1}^6 d_i P_i$. Similarly, we may find generators for the $(1, 1, 3, 1, 1, 3)$ graded piece, for which there are a total of 56. To do so, we may instead find all graphs on 6 vertices with multidegree sequence $(1, 1, 3, 1, 1, 3)$, where an edge connecting the i^{th} vertex to the j^{th} represents a term of the form $\mathbf{A}_i \cdot \mathbf{A}_j$ in the generator. To illustrate this, the example in (A.0.2) would be represented by the graph in Figure A.0.1.

All 56 graphs with multidegree sequence $(1, 1, 3, 1, 1, 3)$ are listed in Figure 3.2.2, and we will

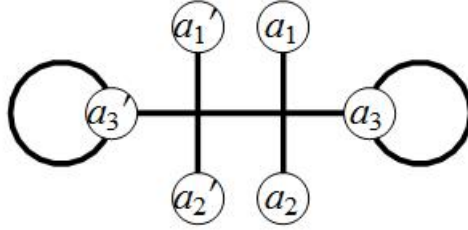


Figure A.0.1:

denote these as $\{\mathbf{B}_i\}_{i=1,\dots,56}$ so that $f(\mathbf{x}) = \sum_{i=1}^{56} c_i \mathbf{B}_i$. Notice that since $I(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3) = -I(\mathbf{a}_2, \mathbf{a}_1, \mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3)$ then any basis element $\mathbf{B}_{j'}$ containing a term of the form $\mathbf{a}_1 \cdot \mathbf{a}_2$ must necessarily have $c_{j'} = 0$. Similarly, $I(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3) = -I(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}'_2, \mathbf{a}'_1, \mathbf{a}'_3)$ implies that any basis element $\mathbf{B}_{j'}$ containing a term of the form $\mathbf{a}'_1 \cdot \mathbf{a}'_2$ must necessarily have $c_{j'} = 0$. In this way we reduce the list of 56 possible basis elements to 42. Moreover, by randomly choosing vectors \mathbf{a}_i , we may generate enough equations in order to find all of the coefficients by numerically calculating $I(A, A')$. In the case when the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3\}$ are taken to be in \mathbb{R}^4 , and if we take $d_1 = -d_3 = 1$ and $d_2 = -d_4 = 3$, then this method easily recovers 3.1.6 numerically. It would be interesting to conjecture what the form of a function like $I_{\langle \text{Lk}^4 \rangle}(A_1, A_2, A_3, A_4)$ might be, and then to use the aforementioned numerical method to approximate it.

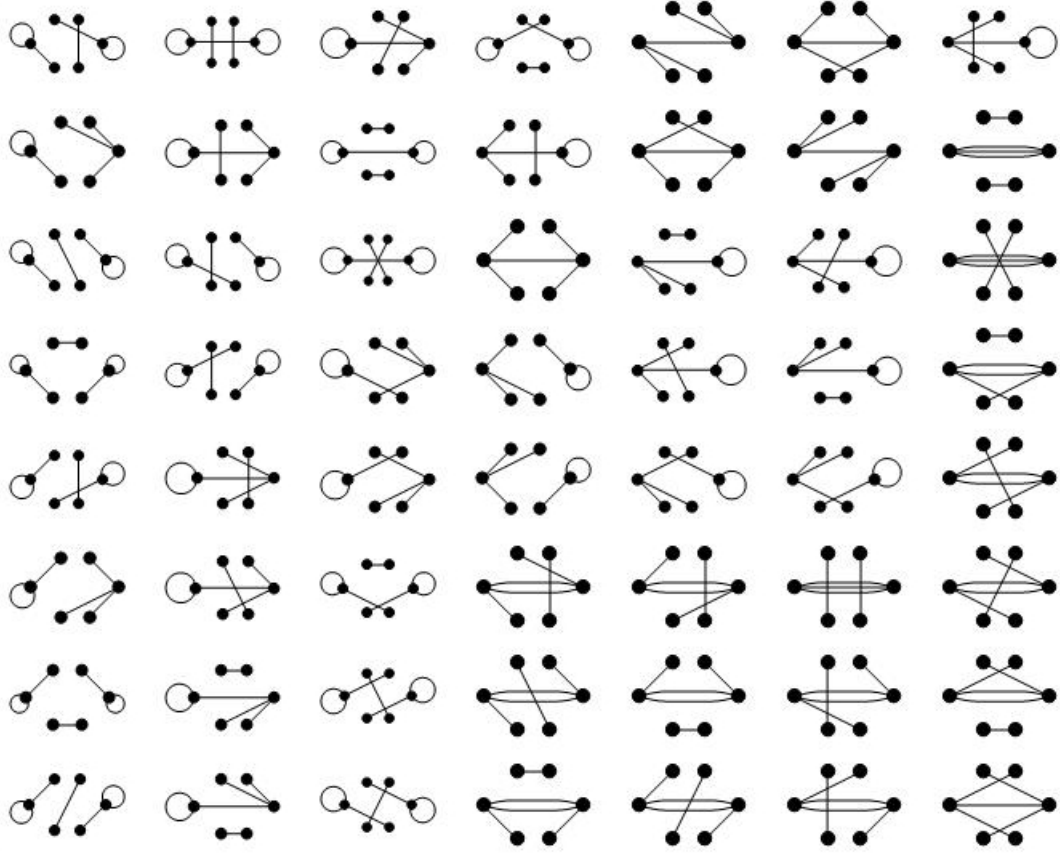


Figure A.0.2: Representatives of the monomial generators of the $(1, 1, 3, 1, 1, 3)$ graded piece. The left side of each graph represents the vectors $\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3$ and the right side represents the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, as in Figure 3.2.1

Appendix B

Petal Diagrams (Python Implementation)

In this appendix we provide some Python code to generate the initial spaces curves in Chapter 5. First we have the definitions of the two initial space curves in \mathbb{R}^{2+2k} where k is the number of petals per component:

```
import numpy as np
def line(t,t1,t2,a,b):
    return (1/(t2-t1))*((t2-t)*a+(t-t1)*b)
def inputcurve1(x,n,eps):
#Petal diagram in first two coordinates:
    listing=[np.cos(n*x)*np.cos(x),np.cos(n*x)*np.sin(x)]
    for i in range(0,n):
        listing.append(np.piecewise(x, [x>=(np.pi/(2*n))*(2*i) and
x < (np.pi/(2*n))*(2*i+1), x>= (np.pi/(2*n))*(2*i+1) and
x < (np.pi/(2*n))*(2*i+2)], [lambda t:
line(t,(np.pi/(2*n))*(2*i), (np.pi/(2*n))*(2*i+1),0,eps),
lambda t: line(t,(np.pi/(2*n))*(2*i+1),
(np.pi/(2*n))*(2*i+2),eps,0)]))
    for i in range(0,n):
        listing.append(0)
    return np.array(listing)
def inputcurve2(x,n,eps):
#Rotated petal diagram in first two coordinates:
    listing=[(np.cos((n-2)*np.pi/n))*np.cos(n*x)*np.cos(x)-
(np.sin((n-2)*np.pi/n))*np.cos(n*x)*np.sin(x),
(np.sin((n-2)*np.pi/n))*np.cos(n*x)*np.cos(x)+
(np.cos((n-2)*np.pi/n))*np.cos(n*x)*np.sin(x)]
    for i in range(0,n):
        listing.append(0)
    for i in range(0,n):
        listing.append(np.piecewise(x, [x>=(np.pi/(2*n))*(2*i)
and x < (np.pi/(2*n))*(2*i+1), x>= (np.pi/(2*n))*(2*i+1)
and x < (np.pi/(2*n))*(2*i+2)], [lambda t:
```



```

        line(t,(np.pi/(2*n))*(2*i), (np.pi/(2*n))*(2*i+1),0,eps),
        lambda t: line(t,(np.pi/(2*n))*(2*i+1),
        (np.pi/(2*n))*(2*i+2),eps,0]))
    return np.array(listing)

```

Once the initial space curves are defined, then we can sample numerous random links by sampling matrices $U \in \text{Mat}_{n,3}$ with $N(0,1)$ entries. In what follows, we sample the n Gaussian matrices, U , make the columns orthonormal by applying the Gram-Schmidt algorithm to columns to generate the new matrix Q , and then we project the initial space curves on to the span of the columns of Q to get a link in \mathbb{R}^3 . We then use the algorithm in [3] to compute the linking number. The second moment of the linking number for these n samples is returned.

```

#Algorithm for computing the linking number in [3]
def linking(v1,v2):
    aa=len(v1);
    bb=len(v2);
    II=0;
    def solidangle(a,b,c):
        return 2*math.atan2(np.linalg.det([a,b,c]),
        np.linalg.norm(a)*np.linalg.norm(b)*
        np.linalg.norm(c)+ np.dot(a,b)*np.linalg.norm(c)+
        np.dot(c,a)*np.linalg.norm(b)+
        np.dot(b,c)*np.linalg.norm(a))
    for i in range(0,aa):
        for j in range(0,bb):
            a=np.array(v2[j])-np.array(v1[i])
            b=np.array(v2[j])-np.array(v1[(i+1)%aa])
            c=np.array(v2[(j+1)%bb])-np.array(v1[(i+1)%aa])
            d=np.array(v2[(j+1)%bb])-np.array(v1[i])
            II=II+solidangle(a,b,c)+solidangle(c,d,a)
    return II/(4*np.pi)
#Gram-Schmidt for triples of vectors
def GS(U):
    q1=U[0]/np.linalg.norm(U[0])
    q2=(U[1]-np.dot(U[1],q1)*q1)/np.linalg.norm(U[1]-np.dot(U[1],q1)*q1)
    q3=(U[2]-(np.dot(U[2],q1))*q1-(np.dot(U[2],q2))*q2)/np.linalg.norm(U[2]
        -(np.dot(U[2],q1))*q1-(np.dot(U[2],q2))*q2)
    return [q1,q2,q3]
#In what follows, n is the number of samples, k is the number of petals
def samples(n,k):
    LinkingNumberList=[]
    for i in range(0,n):
        UU=np.transpose(np.random.normal(0,1,(2*k+2,3)))
        Q=GS(UU)
        #here we set epsilon=1
        def R11(t):
            return np.dot(Q,inputcurve1(t,k,1))

```

```

def R22(t):
    return np.dot(Q, inputcurve2(t,k,1))
vectorlist1=[]
vectorlist2=[]
#Samples 20 points from each component of link
#to feed in to the linking function above
for i in range(0,20):
    vectorlist1.append(R11((np.pi/20)*i))
    vectorlist2.append(R22((np.pi/20)*i))
    LinkingNumberList.append(linking(vectorlist1,vectorlist2))
return (np.average(LinkingNumberList)**2+np.std(LinkingNumberList)**2)

As an example calculation, we ran samples(10000,3) (that is, we sampled 10000 orthonormal
frames and projected the initial space curves corresponding to  $k = 3$  petals per component)
to get that  $\langle Lk^2 \rangle \approx .7181$ . Now compare it to numerically integrating the result in 1.2.3:

#Numerical derivatives, n is number of components
#of vector function being differentiated:
def derivativen(f,n):
    def df(x, h=0.1e-8):
        lists=[]
        for i in range(0,n):
            lists.append(( f(x+h/2)[i] - f(x-h/2)[i] )/h)
        return lists
    return df
def r1(t):
    return inputcurve1(t,3,1)
def r2(t):
    return inputcurve2(t,3,1)
dr1 = derivativen(r1,8)
dr2 = derivativen(r2,8)
def r3(s,t):
    return np.subtract(r2(t),r1(s))
def integrandLK2(a1,a2,a3,a11,a22,a33):
    return (2.0/np.pi)*(((np.dot(a3,a3))*(np.dot(a33,a33))-
        np.dot(a3,a33)**2.0)*np.linalg.det(np.dot([a1,a2,a3],
        np.transpose([a11,a22,a33])))+np.dot(a3,a33)*
        np.linalg.det(np.dot([a3,a11,a22,a33],
        np.transpose([a33,a1,a2,a3]))))/(((np.dot(a3,a3))*
        (np.dot(a33,a33))*((np.dot(a3,a3))*
        (np.dot(a33,a33))-np.dot(a3,a33)**2.0))**(1.5))
def inta(s,t,ss,tt):
    return (1/(16.0*np.pi**2.0))*integrandLK2(dr1(s),dr2(t),r3(s,t),
        dr1(ss),dr2(tt),r3(ss,tt))

#The standard MonteCarlo integration, here n
#is the number of samples from the integration domain
def mcintegrate(n):
    valuelist=[]
    for i in range(0,n):

```

```

s=random.uniform(0,np.pi)
t=random.uniform(0,np.pi)
ss=random.uniform(0,np.pi)
tt=random.uniform(0,np.pi)
valuelist.append(inta(s,t,ss,tt))
#Returns approximation of integral along with an error estimate
return [(np.pi**4)*np.average(valuelist),
        (np.pi**4)*(((np.dot(valuelist,valuelist)/n)-
        (np.average(valuelist))**2)/n)**(0.5)]

```

Running mcintegrate(500000) here we obtain $\langle Lk^2 \rangle \approx .7525$ with an error estimate of .017

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